Theory of Abelian Functions
Part I

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Footnotes are mine, unless marked as Riemann's.

PRELIMINARIES

I. General hypotheses and methods for the study of functions of unbounded variables.

The intent of presenting to readers of the *Journal of Mathematics* my studies on diverse transcendentals and in particular also on Abelian functions, inspires in me the desire, to avoid repetition, to reunite in the commencement, in a preliminary exposition, the general principles which will be the point of departure for my treatment of the subject.

For the independent variable I will take, at first, the geometric representation of Gauss, well-known today, in which a complex variable $z = x + iy$ is regarded as a point on the infinite plane with rectangular coordinates x and y.

I will designate complex numbers and the points which represent them by the same letters. I will consider as a function of $x + yi$ any magnitude w which varies with this quantity while always satisfying the equation

$$i\frac{\partial w}{\partial x} = \frac{\partial w}{\partial y}$$

without hypothesizing an expression for w in terms of x and y. As a consequence of this differential equation, by virtue of a known theorem, the magnitude w is representable by a series proceeding following the integral powers of $z - a$, of the form

$$\sum_{n=0}^{n=\infty} a_n (z-a)^n,$$ [*]

provided that in the neighborhood of a it admits everywhere of *one* determined value, varying continuously with z; and that this possibility of representation holds up to a distance from a, which is to say a value of mod. $(z - a)$, where a discontinuity is found.

With the aid of considerations, which rest upon the principles of the method of undetermined coefficients, we recognize that the coefficients a_n are completely determined when w is given along a finite line from a, however small it may be.

In uniting these two propositions, one can be easily assured of the exactitude of this theorem:

A function of x+yi, which is given in a portion of the (x,y) plane, can be prolonged beyond it in a continuous manner in only one way.

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[*] This is what gets called a "Taylor expansion" or a "Taylor series" these days, although it's Leibniz's idea, not Taylor's.

Now, let us conceive of the function we are considering as not determined by any expressions or analytic equations containing z, but rather by the fact that the value of the function is given in a portion of the z plane with arbitrary boundaries, and that it is extended in a continuous manner by means of the partial differential equation

$$i\frac{\partial w}{\partial x}=\frac{\partial w}{\partial y}.$$

This extension performed by the power of the preceding propositions, is completely determined, if it is performed not along pure lines (for then the application of a differential equation is impossible), but along surface bands of finite width. Now, according to the nature of the extended function, it either will, or will not always have the same value for the same value of z, whatever may be the path along which this extension was performed.

In the first case, I will name it *uniform*; it would then be perfectly determined for any value of z and will never be discontinuous along a line. In the second case, where it is called *multiform,* attention must be given to certain points on the z plane to understand the development of the function, points around which the function extends into another. For example, the point a for the function log $(z - a)$, is such a point.

Let us conceive a line drawn from this point a; we can, in the vicinity of a, choose the value of the function, such that the function varies in a continuous manner except along this line; but, on the two sides of this line, it takes different values, the value on the negative side being $2\pi i$ greater than the value on the positive side.[1]

The extension of the function beyond one of the sides of this line, e.g. the negative side, in the region located on the other side, gives therefore evidently a different function than that which appeared formerly, and which, in the case envisaged here, exceeds everywhere this latter by $2\pi i$.

To simplify the designations of these relations, we will name the various extensions of *one* function on the same portion of the z plane the *branches* of this function, and a point from which one branch of the function extends into another a *branch-point* of the function. Wherever there is no branching, the function is considered as *monodrome* or *uniform.*

A branch of a function of multiple independent variables z, s, t, … is *uniform* in the vicinity of a determined system of values $z=a$, $s=b$, $t=c$, … when to all combinations of values up to a finite distance from this one (which is to say when a determined finite magnitude of moduli $z - a$, $s - b$, $t - c$, …) correspond a determined value of this branch of the function, varying in a continuous manner with the variables.

Branch-points, or points around which one branch extends into another, are formed, for a function of multiple variables, by all the values of the independent variables satisfying an equation among these variables. [There exists an equation written in terms of the variables, whose solutions give the branch points – JAR]

Following a known theorem, of which we have spoken earlier, the property of a function being uniform comes down to the question of its susceptibility of being developed by positive or negative integral powers of growing variables, and the

[1] Following the designation of Gauss, who names $+i$ the positive lateral unity, I call the positive side of a given line that side which is situated in regards to the direction of the line in the same way as $+i$ is situated in regards to 1. —RIEMANN

branching of a function comes down to the impossibility of such a development. But it does not seem useful to express properties that are independent of the mode of representation, by the help of characters that depend on an explicit determined form of the expression of the function.

In several studies, notably in the study of algebraic and Abelian functions, it will be useful to represent the mode of branching of a multiform function in the following geometric fashion:

Let us conceive a surface extended on the (x,y) plane and coinciding with it [with the plane] (or if you like, an infinitely thin body extended on the plane), which stretches as far and only as far as the function is given for the plane. When the function is extended, this surface will thus be extended just as far. In a region of the plane where two or more extensions of this function are found, the surface will be double or multiple. It will be composed of two or more leaves each corresponding to a branch of the function. Around a branch-point of the function, one leaf of the surface will extend into another leaf, and in such a way that, in the vicinity of this point, the surface can be regarded as like a helicoid whose axis is perpendicular to the (x,y) plane in this point and with infinitely small screw threading. But when the function, after z has made several rotations around the branch-point, takes again its initial value [as, for example, $(z - a)^{m/n}$, m, n being relatively prime, after n rotations around a], one will have to suppose then that the leaf above the surface joins the lower leaf in passing through the rest of the leaves.

The multiform function has for each branch-point, *one single* determined value, and can thus be regarded as a perfectly determined function of location (at a point) on this surface.

II. Theorems of "Analysis Situs" relative to the theory of integrals of total differentials having two terms.

In the study of functions which result from the integration of total differentials, some theorems pertaining to *Analysis Situs* are almost indispensable. Under this designation employed by Leibniz, although in a slightly different sense, we can order a part of the study of continuous magnitudes where we do not consider these magnitudes as existing independently of their position and as measurable by each other, but where one studies only the relationships of situation of locations and regions, in completely abstracting all metric relationships.

Since I have the intention, on another occasion, to treat this subject which completely abstracts metric relationships, I will content myself to express in geometric form some necessary theorems for the integration of total differentials of two terms.

Let T be a given surface, covering one or many times the (x,y) plane, and let X, Y be continuous functions of location on this surface, such that Xdx + Ydy are everywhere total differentials, which is to say that we have everywhere

$$\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x} = 0.$$

We know then that

$$\int (X\,dx + Y\,dy),$$

the integral being taken around a part of the surface T in the positive or negative direction – that is to say all around the boundary, either always in the positive sense or always in the negative sense, in regard to the exterior normal (compare to the footnote in §1) – is zero, since in the first case this integral is identical to the surface integral

$$\int\left(\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y}\right)dT,$$

relative to this part of the surface, and, in the second case, is equal to the above equation with changed sign.

Consequently, the integral

$$\int(X\,dx + Y\,dy),$$

taken between two fixed points, along two different paths, has the same value when the joining of these paths forms the complete bounding of a part of surface T. It follows, that when in the interior of T, any curve which closes itself by coming back upon itself makes the complete boundary of a part of T, the integral taken from a fixed initial point to the same final point has always the same value, and is a continuous function of the position of the final point everywhere on T and independent of the path of integration.

This gives rise to a distinction between simply connected surfaces, where each closed curve completely encloses a portion of surface (as, for example, a circle), and a multiply connected surface where this is not the case (as, for example, the ring whose boundary is formed by two concentric circumferences). A multiply connected surface can be transformed, by the effect of cuts, into a simply connected surface. (See, for example, the figures at the end of this section.)

Since this operation renders important services in the study of integrals of algebraic functions, we will here briefly present the propositions which are treated there; they are valuable for surfaces situated in any fashion in space.

When on a surface F two joined systems of curves *a* and *b* form the complete boundary of a part of this surface, any other system of curves which, joined with *a* forms the complete boundary of a part of F, forms also, when it is joined with *b*, the boundary of a part of the surface, which is composed then of the first two surface portions situated along *a* (and this by addition or by subtraction, depending on whether these two parts are not situated on the same side of *a* or if indeed they are).

The two systems of curves play, therefore, the same role relative to the complete bounding of a part of F, and can thus each replace the other for this goal[1].

When on a surface F one can draw n *closed curves* $a_1, a_2, \ldots a_n$ *which, whether they are considered separately or are considered as joined, do not form a complete boundary of a portion of this surface, but which, joined with any other closed curve, then form the complete boundary of a portion of surface, the surface will be called (n+1)-ply connected.*

This characteristic of the surface is independent of the choice of the system of curves $a_1, a_2, \ldots, a_n$ since n other closed curves $b_1, b_2, \ldots, b_n$, which do not suffice to form the complete boundary of a surface portion, will also totally bound a part of F, if we combine them with any other closed curve.

Indeed, since b_1, reunited with the lines *a*, completely bounds a part of F, one of these curves *a* can be replaced by b_1 and the remaining *a* curves. Consequently, the combination of b_1 with these $n-1$ curves *a* can be replaced by b_1, b_2 and the $n-2$ remaining *a* curves. When, as is supposed here, the curves *b* do not suffice to form the complete boundary of a portion of the surface F, this process can clearly be continued until all the *a* curves be replaced by *b* curves.

An (n + 1)-ply connected surface F *can be transformed, by the effect of a cross-cut [Querschnitt], that is to say a cut emanating from a point on the boundary, crossing the interior of the surface and coming out at another point on the boundary, into an n-ply connected surface* F'. *The sections of the boundary, so long as they are created by the effect of this cut, play the role of boundary throughout this operation, such that a cross-cut can not cross any point on the surface more than once, but can terminate in a point of its own previous path.*

Since the lines $a_1, a_2, \ldots, a_n$ do not suffice to form the complete boundary of a part of F, it is necessary, when we conceive of F as cut into sections by these lines, as well as the surface portion situated along the left side of a_n, encloses again a distinct part of the boundary of lines *a* and, consequently, forming part of the boundary of F. [[Previous sentence is poorly translated; the meaning may be off.]] One can then, from a point of a_n and across any of these sections just as well as any other, create a section that does not cut any of the *a* curves and ending at the boundary of F. These two lines *q'* and *q"* form then, by their combination, a cross-cut *q* of the surface F, which section fulfills our goal.

Indeed, let us consider as F', the surface into which F is decomposed by the effect of this cross-cut; the lines $a_1, a_2, \ldots, a_{n-1}$ have their path in the interior of F' and are closed curves which do not suffice to form the bounding of a part of F, or a part of F' either. But any other closed curve *l*, having its pathin the interior of F', forms then with these lines the complete bounding of a portion of F'. Indeed, the connection of line *l* with a system of lines $a_1, a_2, \ldots, a_n$ forms the complete bounding of a portion *f* of F. Now, it can be demonstrated that, in this last bounding, a_n cannot present itself; in effect, if it could, when *f* is situated on the left side or the right side of a_n, then *q'* or *q"* would traverse *f* to end at a boundary point of F, which is to say at a point which is not part of *f* and, it follows, would cut *f*, which would be contrary to the hypothesis that *l* as well as the lines *a*, excepting the point where a_n and *q* cut each other, are always situated inside F'.

The surface F', resulting from the decomposition of F by the cross-cut *q*, is therefore, as we have said, an *n*-ply connected surface.

It now concerns us to demonstrate that the surface F, by the effect of any cross-cut *q*, which does not decompose it into separate portions, is decomposed into an *n*-ply connected surface F'. When the surface portions, situated on both sides of the cross-cut *p*, are connected – that is to say that they are not separated – we can draw across F' a line *b*, beginning on a side of the cross-cut to end at the same point on the opposite side.

This line *b* forms therefore in the interior of F a closed curve coming back on itself and which, since the cross-cut goes from one side to the other of this line at a boundary point, cannot form the complete bounding of either of the two surface portions into which it cuts F. One can thus replace one of the *a* curves by the curve *b*, and each of the *n* – 1 remaining *a* curves by a curve in the interior of F' and, moreover, if this is necessary, by the curve *b*; which, by using the previous reasoning, permits us to say that F' is *n*-ply connected.

To make these considerations applicable to an unbounded surface – that is, a closed surface – we must transform this closed surface into a bounded surface, by making a hole in any point such that the first decomposition will be effected by means of a cross-cut beginning at this point and returning to it, and consequently forming a closed curve.

For example, the exterior surface of a torus, a triply connected surface, will be transformed into a simply connected surface by means of a closed curve and a cross-cut.

We will now apply the decomposition of multiply connected surfaces into simply connected surfaces by the consideration of the integral of the exact differential

$$X\,dx + Y\,dy,$$

treated at the beginning of the present § II. When the surface T, which covers the (x, y) plane, and on which X, Y are continuous functions of position, satisfying the equation

$$\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x} = 0,$$

is n-ply connected, we will decompose it into a simply connected surface T' by making n cross-cuts.

Then the integration X dx + Y dy, taken from a fixed initial point along the curves situated in the interior of T', gives a value which depends only on the position of the final point and which can be regarded as a function of the coordinates of this point.

Substituting for these coordinates the magnitudes x, y, we obtain z, a function of x, y, where

$$z = \int (X\,dx + Y\,dy)$$

is completely determined for any point of T' and varies continuously everywhere in the interior of T', but which, in crossing a cross-cut, varies in general by a finite constant value along the line which leads from a node of the network of sections, to another node.

The variations in the crossing of cross-cuts depend on independent magnitudes whose number is equal to the number of cross-cuts; indeed, if one runs through the network of sections in the reverse direction, each cross-cut before being traversed in beginning at its final extremity, each variation is everywhere determined, when we are given the value of the origin of the cross-cut; but these values are mutually independent.

To give a more intuitive representation of what is meant by n-ply connected surfaces which we have earlier defined, we will present here as examples, figures of simply, doubly, and triply connected surfaces.

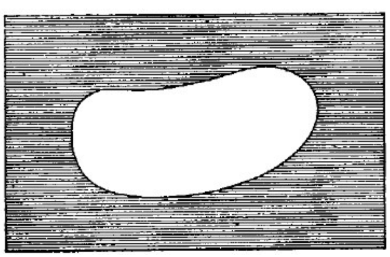

Fig. 4.

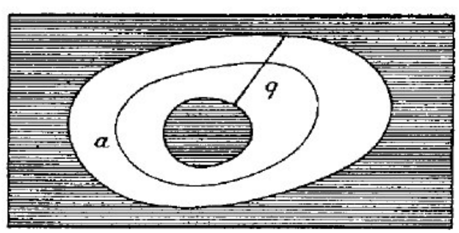

Fig. 5.

Simply Connected Surface

Any cross-cut which one makes, divides this surface (fig. 4), and any closed curve which is drawn within it forms the complete boundary of a part of the surface.

Doubly Connected Surface

This surface (fig. 5) is decomposed into a simply connected surface by any cross-cut q which does not divide it.

Any closed curve, can form, with the addition of curve *a*, the complete boundary of a portion of the surface. [*Any* closed curve, will, in combination with *a*, *definitely* form the complete boundary of a surface portion.]

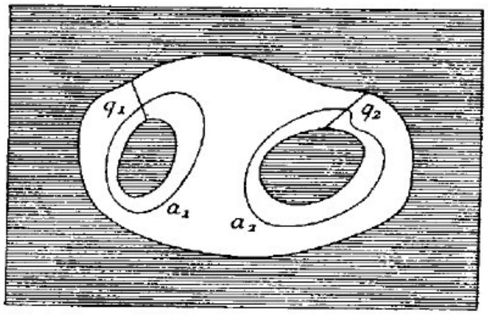

Fig. 6.

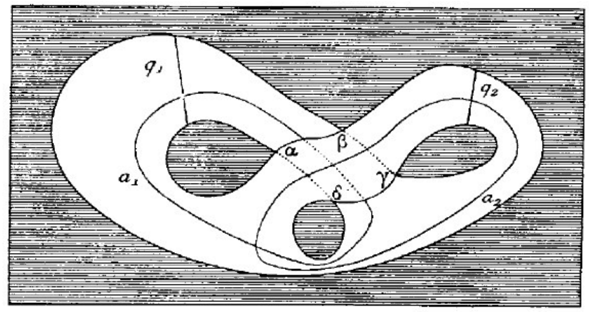

Fig. 7.

Triply Connected Surface

On this surface (fig. 6 or 7) any closed curve can, with the addition of the curves a_1 and a_2, form the complete boundary of a portion of the surface. It is decomposed by any cross-cut which does not divide it, into a doubly connected surface, and two similar cross-cuts q_1 and q_2 decompose it into a simply connected surface.

In the portion αβγδ the plane is doubly covered by the surface. In this portion, the leaf of the surface on which a_1 makes its way must be considered as passing below the other – as is indicated by the hashing of the lines in that portion.

[p101]

III. Determination of a function of a complex variable by the conditions which it fulfills regarding boundary and discontinuities.

On a plane, where the rectangular coordinates of a point are *x*, *y*, when the value of a function of $x + yi$ is given along a finite line [not necessarily a "straight" line], this function can be further extended in a continuous fashion in only one unique way, and, consequently, it is by this still perfectly determined. (*see page 1 or 2*). But neither can it be taken arbitrarily along this line, as it must be susceptible of a continuous extension on the portions of surface situated on the two sides of this line, since, by its very progress along a finite part, however small it be, it is already determined for the remaining part. Thus, in the mode of determination of a function, the conditions which serve for this determination are not mutually independent.

As the foundation principle in the studies of transcendentals, it is necessary above all else to establish a system of mutually independent conditions, sufficing to determine the function. Here, in many cases, and notably among these: the integrals of algebraic equations and of their inverse functions, one can make use of a principle by the help of which Dirichlet – probably inspired by a similar thought of Gauss[*] – treated for a number of years in his Lectures on forces which act by the inverse square of their distance, the solution of this problem relative to a function of three variables satisfying Laplace's

[*] See http://www.wlym.com/~jross for Gauss's paper on forces acting by the inverse square of their distance.

partial differential equation. There exist, however, in this application to the theory of transcendentals, a particularly important case where one cannot make use of this principle in the simplest form given by Dirichlet in his Lessons where this case can be neglected as being of minor importance; this case is that where the function, in certain points of the domain where it is to be determined, must admit stipulated discontinuities; by this we must mean that in every one of these points the function is subjected to the condition of being discontinuous, as though it is an assigned discontinuous function in these points, or must differ from being continuous only in these points. I will present the principle in the necessary form for the application in view, and I will take the liberty to refer the reader, with regard to certain supplementary studies, to the explanations that I published on this question in my Doctoral Dissertation, where I exposed this principle. (*Inaugural Dissertation: §XVII.*)[(*)]

Let there be a surface T with arbitrary boundary covering the (x,y) plane one or more times, and, on this surface, two real functions of x, y, namely α and β, which are uniformly determined for each of the points of the surface; and designate by $\Omega(\alpha)$ the integral

$$\int\left[\left(\frac{\partial\alpha}{\partial x}-\frac{\partial\beta}{\partial y}\right)^2+\left(\frac{\partial\alpha}{\partial y}+\frac{\partial\beta}{\partial x}\right)^2\right]dT,$$

relative to the surface T, the functions α and β admitting of arbitrary discontinuities, provided that the integral does not thereby become infinite. Then $\Omega(\alpha - \lambda)$ also remains finite, when λ is everywhere continuous and everywhere admits a finite derivative. If this continuous function λ is submitted to the condition of being different from a discontinuous function γ in only an infinitely small part of the surface T, then $\Omega(\alpha - \lambda)$ will become infinitely large, when γ is discontinuous along a line or when it is discontinuous in a point in such a way that

$$\int\left[\left(\frac{\partial\gamma}{\partial x}\right)^2+\left(\frac{\partial\gamma}{\partial y}\right)^2\right]dT$$

be infinitely large (*Inaugural Dissertation §XVII*). But $\Omega(\alpha - \lambda)$ will remain finite, when γ is discontinuous solely in isolated points and that only in such a way that the integral

$$\int\left[\left(\frac{\partial\gamma}{\partial x}\right)^2+\left(\frac{\partial\gamma}{\partial y}\right)^2\right]dT,$$

taken across surface T, remains finite, as, for example, when γ, in the neighborhood of a point, is, up to a distance r from this point, equal to $(-\log r)^{\varepsilon}$ when $0 < \varepsilon < ½$. To speak more briefly: the functions which can represent the function λ, without $\Omega(\alpha - \lambda)$ ceasing to be finite, are called *discontinuous of the first species*, while functions for which this is not possible will be called *discontinuous of the second species*. If one conceives then that in $\Omega(\alpha - \mu)$ one takes for μ all functions, continuous or discontinuous of the first species which vanish on the boundary, the integral will always take a finite value and, by its very nature, will never be negative; it is necessary then that for $\alpha - \mu = u$ there is at least one

(*) This Inaugural Dissertation is his 1851 *Theory of Complex Functions*.

minimum value, such that Ω, for any function $\alpha - \mu$ which differs infinitely little from u, will be greater than $\Omega(u)$.

Let us then designate by σ an arbitrary function of location on the surface T, continuous or discontinuous of the first species, and everywhere equal to zero on the boundary, and designate by h a magnitude independent of x, y; it is necessary that $\Omega(u + h\sigma)$ be greater than $\Omega(u)$, for a value sufficiently small of h, either positive or negative; such that, in the development of this expression by powers of h, the coefficients of h disappear. In this case, we have

$$\Omega(u + h\sigma) = \Omega(u) + h^2 \int \left[\left(\frac{\partial \sigma}{\partial x} \right)^2 + \left(\frac{\partial \sigma}{\partial y} \right)^2 \right] dT,$$

and, consequently, Ω is always a minimum. The minimum is found only for a unique function u; indeed, if a minimum exists also for $u + \sigma$, we cannot have

$$\Omega(u + \sigma) > \Omega(u);$$

otherwise we would have

$$\Omega(u + h\sigma) < \Omega(u + \sigma)$$

for $h < 1$; $\Omega(u + \sigma)$ consequently could not be smaller than the function Ω for the values in the neighborhood of $u + \sigma$. But if $\Omega(u + \sigma) = \Omega(u)$, σ must be a constant, and since it is zero on the boundary, it must be zero throughout. It is thus only for a unique function u that the integral Ω is a minimum and that the variation of the first order or the term proportional to h in $\Omega(u + h\sigma)$,

$$2h \int dT \left[\left(\frac{\partial u}{\partial x} - \frac{\partial \beta}{\partial y} \right) \frac{\partial \sigma}{\partial x} + \left(\frac{\partial u}{\partial y} + \frac{\partial \beta}{\partial x} \right) \frac{\partial \sigma}{\partial y} \right] = 0.$$

It results from this last proposition that the integral

$$\int \left[\left(\frac{\partial \beta}{\partial x} + \frac{\partial u}{\partial y} \right) dx + \left(\frac{\partial \beta}{\partial y} - \frac{\partial u}{\partial x} \right) dy \right],$$

taken on the complete boundary of a portion of surface T, is always zero.

Now, if we decompose the surface T (following the preceding method), when it is a multiply connected surface, into a simply connected surface T', the integration taken in the interior of T' from an initial fixed point to point (x,y) gives a function of x, y,

$$v = \int \left[\left(\frac{\partial \beta}{\partial x} + \frac{\partial u}{\partial y} \right) dx + \left(\frac{\partial \beta}{\partial y} - \frac{\partial u}{\partial x} \right) dy \right] + \text{const.},$$

which is everywhere either continuous or discontinuous of the first species on T', and which, in crossing the cross-cuts changes by finite values which are constant between two knot-points of the network of sections. Thus $v = \beta - \nu$ (vee equals beta minus nu) satisfies the equations

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \qquad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

and, consequently, $u + iv$ is a solution of the differential equation

$$\frac{\partial (u + vi)}{\partial y} - i \frac{\partial (u + vi)}{\partial x} = 0,$$

which is to say it is a function of $x + yi$.

We thus obtain the following theorem, announced in §XVIII of the above-cited Memoir:

When on a connected surface T, *decomposed by cross-cuts into a simply connected surface* T*', we give a complex function of x,y, namely* $\alpha + \beta i$, *for which the integral*

$$\int\left[\left(\frac{\partial\alpha}{\partial x}-\frac{\partial\beta}{\partial y}\right)^2+\left(\frac{\partial\alpha}{\partial y}+\frac{\partial\beta}{\partial x}\right)^2\right]dT,$$

extended over the entire surface, takes a finite value, this function can always, and in a unique way, be transformed into a function of x + yi by the subtraction of a function $\mu + \nu i$ of *x,y, satisfying the following conditions:*

1. *On the boundary* $\mu = 0$, *or at least only differing from* 0 *at isolated points; in a point,* ν *is given in an arbitrary fashion.*
2. *The variations of* μ *on* T, *and those of* ν *on* T*' are only discontinuous in isolated points, and that only in such a way that the integrals*

$$\int\left[\left(\frac{\partial\mu}{\partial x}\right)^2+\left(\frac{\partial\mu}{\partial y}\right)^2\right]dT \quad \text{and} \quad \int\left[\left(\frac{\partial\nu}{\partial x}\right)^2+\left(\frac{\partial\nu}{\partial y}\right)^2\right]dT$$

taken along the whole surface remain finite; as well as having the various values of ν *along a cross-cut being equal on the two sides.*

When the function $\alpha + \beta i$, in the points where its derivatives become infinite, is discontinuous like a function of *x + yi* in these points, and when the function has no discontinuities which disappear by a modification of its value in an isolated point, then $\Omega(\alpha)$ remains finite and $\mu + \nu i$ is everywhere continuous on T'.

In truth, since a function of *x + yi* can admit of certain discontinuities, only such as for example, the discontinuities of the first species (*Inaugural Dissertation,* § XII), the difference of two similar functions must be continuous, provided that they not be discontinuous of the second species.

Thus, following the demonstrated theorem, a function of *x + yi* can be determined such that in the interior of T, making abstraction of discontinuities of the imaginary part relative to cross-cuts, there are the prescribed discontinuities, and that its real part takes on the boundary a value which is everywhere given arbitrarily on the boundary; which presupposes that, in any point where the derivatives of the function become infinite, the prescribed discontinuity is that of a given function of *x + yi*, discontinuous in this point. [[This paragraph is terribly translated!]]

The condition relative to boundary, can, as is easy to understand, be replaced by numerous others without the conclusions meeting with essential modifications.

IV. Theory of Abelian Functions

In the work which follows, I treat Abelian functions following a method whose principles I explained in my *Inaugural Dissertation*, and which I just now presented in the three preceding sections under a slightly modified form.

For the ease of reading of these studies, I will precede them with a summary account.

The first Section contains the theory of a system of algebraic functions with the same branchings and their integrals, without it being necessary to broach the subject of the theta series. In §§I-V, the determination of these functions by their mode of branching and their discontinuities is treated; from §VI to §X their rational expressions by functions of two variables connected by an algebraic equation is considered; and §XI to §XIII covers the transformation of these expressions by rational substitutions. In this study is offered the conception of a *class* of algebraic equations, which can be mutually transformed into each other by rational substitutions, and which may also be of great importance in other studies, and the transformation of one equation of this nature into equations of its same class but of minimum degree will also be useful in other circumstances. Finally, the final §XIV to §XVI of this Section, as preliminaries to Section II, covers applications of the additive theorem of Abel, relative to an arbitrary system of everywhere-finite integrals of algebraic functions of the same branchings, to the integration of a system of differential equations.

In the second Section, in the case of an arbitrary system of everywhere-finite integrals of equal-branching ($2p$+1)-fold-connected algebraic functions, we will express the Jacobian inverse functions of p variables, by the aide of the theta series p-fold infinite, which is to say by the aid of the series of the form

$$\vartheta(v_1, v_2, \ldots, v_p) = \left(\sum_{-\infty}^{\infty}\right)^p e^{\left(\sum_1^p\right)^2 a_{\mu,\mu'} m_\mu m'_\mu + 2\sum_1^p v_\mu m_\mu}$$

where the summations in the exponent relate to μ and μ', and where the other summations relate to $m_1, m_2, \ldots, m_p$. One recognizes that, for the general solution of this problem, a certain class of theta functions suffices; this class becomes particular for $p > 3$, a condition where, among the $p(p + 1) / 2$ magnitudes a, there hold $(p - 2)(p - 3)/1 \cdot 2$ relations, such that, among these magnitudes, only $3p - 3$ are arbitrary.

This part of the Memoir forms at the same time a theory of this particular species of ϑ functions. General ϑ functions will remain excluded from this study; they can be treated by a completely analogous method.

The inversion problem of Jacobi, resolved here, has already been resolved in several manners in the case of hyperelliptic integrals by the persevering and crowning works of such success of Weierstrass; an insight of this had appeared in Volume 47 of *Crelle's Journal of Mathematics*, page 289. It is, however, not until today the section of his works, expressed in sections § I, II, and the first half of § III pertaining to elliptical functions, of which the detailed explanation was published (*Crelle's Journal*, v. 52, p. 285). The coincidence which can be found in the latter parts of the work of Weierstrass and those which I present here, not only in the results, but also in the methods which lead to them, could not be understood until the time of the publication of these cited works.

The work which follows, with the exception of the two last § XXVI, XXVII, of which the subject can be but briefly indicated in my Lessons, is an analysis of a part of those which I taught at Göttingen from St-Michael's in 1855 to the same day in 1856.

Relative to the discovery of certain results and as for § I-V, IX and XII and as for the preliminary theorems which I expounded later in my Lessons, in the manner expressed in the present work, I was led to them during the autumn of 1851 and the beginning of 1852, by the studies on the conformal representation of multiply connected surfaces; but, later, I was turned away from this research by another subject. I did not take up this work again until Easter of 1855 and it was continued just up to § XXI, during the Easter vacation and the St-Michael vacation of the same year. The rest was completed around St-Michael of 1856.

A certain number of supplementary propositions are presented in numerous locations during the subsequent work.

SECTION I.

§ I.

Let s be the root of an irreducible equation of degree n whose coefficients are integral functions of z of degree m; to each value of z correspond n values of s, which vary with z continuously throughout where they are not infinite. One represents therefore according to § 3, the method of branching of this function, by the in-the-z-plane-extended unbounded surface T, such that it is in each part of the plane n-fold, and s is then a single-valued function of position in this surface.[**]

An unbounded surface can be considered as a surface whose boundary is infinity or as a closed surface, and it is from this point of view that we will regard the surface T, such that the value $z=\infty$ corresponds to *one* point on each of the leaves, unless $z=\infty$ is a branch-point.[*]

Every rational function of s and z clearly is equally a uniform function of position on the surface T, and possesses therefore the same method of branching as the function s. We will see below that the reciprocal is also true.

The integration of such a function gives a function whose different extensions for the same portion of the surface T differ only by constants, since its derivative for the same point of surface takes always the same value.

Such a system of equal-branching algebraic functions and integrals of these functions will be the first object of our study; but, in lieu of starting with the expressions for these functions, we will define them by their discontinuities in applying *Dirichlet's Principle* (p. 104).

§ II.

To simplify the following, I will say that a function is *infinitely small of the first order at a point of the surface* T when its logarithm increases by $2\pi i$, when one moves in the positive direction around the circumference of a portion of this surface which includes this point, where the function remains finite and not zero. This is the case for a point where the surface T turns on itself μ times, when z is equal to a finite value a, for $(z - a)^{1/\mu}$ which is to say $(dz)^{1/\mu}$; but, when $z=\infty$, it is $(1/z)^{1/\mu}$ which is infinitely small of

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[**] The value has n values because there are n roots of an algebraic equation of degree n (the 1799 paper). Here, instead of $x^m + Ax^{m-1} + \ldots$, you have $x^n + (z^m + Az^{m-1} + \ldots)x^{n-1} + \ldots$ So, the coefficients are not constants A, B, C, D, as before, but are instead functions of z.

[*] Making a Riemann sphere instead of a Gauss plane.

the first order. The case where a function becomes infinitely small or infinitely great of order ν at a point of the surface T can be treated as if the function there became infinitely small or infinitely great of the first order at ν points which coincide (or which come indefinitely close to each other). We will find occasion to do this in what follows.

The manner in which the functions that we will treat here, become discontinuous can be explained thus. If one of them is infinite at a point of the surface T, it can always be transformed — by the subtraction of a finite expression of the form

$$A \log r + B r^{-1} + C r^{-2} + \ldots,$$

when r designates an arbitrary function which becomes infinitely small of the first order at this point — into a function which is continuous at this point. This is clear following the known propositions on the development of a function into a series of powers, propositions which one can demonstrate with Cauchy, or with the Fourier series.

§ III.

Let us conceive a connected surface T, completely covering the z plane n times, without boundaries, but which, in keeping with the proceeding, we can consider as a closed surface, and let us conceive it as decomposed into a simply connected surface T'. As the boundary curve of a simply connected surface is formed by a unique curve ((contour)), but that a closed surface takes, by the effect of an odd number of sections, an even number of bounding portions, and, by the effect of an even number of sections, an odd number of bounding portions, to effect the decomposition of the surface, it will therefore be necessary to make use of an even number of sections. Let $2p$ be the number of these cross-cuts. To simplify what will follow, the decomposition will be performed so that each new section be made from a point of one of the preceding sections and bordering the neighboring point on the other side of this same section; so, when a magnitude varies continuously along the entire boundary of the surface T', and having confirmed in the entire system of sections equal variations on both sides, the difference between the two values which it takes at the same point of the network of sections is equal to the same constant in all parts of a single cross-cut.

Consider $z = x + yi$ and consider on T a function $\alpha + \beta i$ of x, y defined as follows:

In the neighborhood of points $\varepsilon_1, \varepsilon_2, \ldots$ it will be determined as equal to the given functions of $x + yi$ which are infinite in these points, in such a way that at ε_ν, it will be equal to a finite expression of the form

$$A_\nu \log r_\nu + B_\nu r_\nu^{-1} + C_\nu r_\nu^{-2} + \ldots = \phi_\nu(r_\nu),$$

where r_ν designates an arbitrary function of z which becomes infinitely small of the first order at ε_ν, and $A_\nu, B_\nu, C_\nu, \ldots$ being arbitrary constants. We will next draw in the interior of T', to any point from each of the points ε with magnitude A not equal to zero, lines which do not cut themselves; the line leaving ε_ν will be designated by l_ν. Finally let us suppose for all of surface T still remaining, that the function will be defined thus: besides the lines l and the cross-cuts, it is everywhere continuous; on the positive (left) side of the line l_ν, it exceeds by $-2\pi i A_\nu$, and on the positive side of the ν^{th} cross-cut by the given constant $h^{(\nu)}$ the respective values which it has on the sides opposite these cuts; finally the integral

$$\int\left[\left(\frac{\partial\alpha}{\partial x}-\frac{\partial\beta}{\partial y}\right)^2+\left(\frac{\partial\alpha}{\partial y}-\frac{\partial\beta}{\partial x}\right)^2\right]dT\ ,$$

taken over the surface T, has a finite value. It is easy to understand that this is always possible when the sum of all the magnitudes A is equal to zero, and further is *only* possible in this condition, for it is only in this case that the function, after making a circuit along the system of lines l, can retake anew its preceding value.

The constants of addition $h^{(1)}$, $h^{(2)}$, … $h^{(2p)}$, by which a like function grows in passing from the negative to the positive side of the cross-cuts, are called the *moduli of periodicity* of this function.

Now, following *Dirichlet's principle*, the function $\alpha + \beta i$, can be transformed into a function ω of $x + yi$ by the subtraction of a like function of x, y everywhere continuous in T', with purely imaginary moduli of periodicity, and this function is completely determined by a nearby additive constant. The function ω admits of the same discontinuities as $\alpha + \beta i$ in the interior of T' and the same real parts as the moduli of periodicity. Consequently, in the composition of ω, the functions ϕ_ν and the real parts of the moduli of periodicity can be given arbitrarily. In regard to these conditions, the function is completely determined by a nearby additive constant and, consequently, it is likewise with the imaginary part of these moduli of periodicity. We will see that this function ω subsumes all the functions indicated in § I.

§ IV.

Everywhere-finite functions ω (integral of the first sort)

We will first of all consider the simplest among these functions, those which remain always finite and which, consequently, are continuous throughout the interior of the surface T'. If we designate by w_1, w_2, …, w_p such functions, we have also, as a function of the same sort,

$$w = \alpha_1 w_1 + \alpha_2 w_2 + \ldots + \alpha_p w_p + \text{const.},$$

α_1, α_2, …, α_p being arbitrary constants. Let us designate the moduli of periodicity of the functions w_1, w_2, …, w_p relative to the ν^{th} cross-cut by $k_1^{(\nu)}$, $k_2^{(\nu)}$, …, $k_p^{(\nu)}$. The modulus of periodicity of w relative to this cross-cut is then

$$\alpha_1 k_1^{(\nu)} + \alpha_2 k_2^{(\nu)} + \cdots + \alpha_p k_p^{(\nu)} = k^{(\nu)};$$

and, if we write the magnitudes α in the form $\gamma + \delta i$, the real components of the $2p$ magnitudes $k^{(1)}$, $k^{(2)}$, … $k^{(2p)}$ are linear functions of the magnitudes γ_1, γ_2, …, γ_p, δ_1, δ_2, …, δ_p.

Now, when no linear equation of constant coefficients exists among the magnitudes w_1, w_2, …, w_p, the determinant of these linear expressions cannot disappear.

Indeed, if it were not thus the case, one would be able to determine the relationships among the magnitudes α in such a way that the moduli of periodicity of the real part of w would all be equal to zero, and that, consequently, the real part of w, and consequently also w itself must, by virtue of Dirichlet's principle, be reduced to a constant.

It follows that the $2p$ magnitudes γ and δ can be determined in such a way that the real parts of the moduli of periodicity take given values; consequently, w can represent any function ω remaining always finite, when $w_1, w_2, \ldots, w_p$ satisfy no linear equation of constant coefficients. But these functions can always be chosen in a manner that satisfies this condition; indeed, as long as $\mu < p$, equations of linear conditionality hold among the moduli of periodicity of the real part of

$$\alpha_1 w_1 + \alpha_2 w_2 + \ldots + \alpha_p w_p + \text{const.},$$

and thus $w_{\mu+1}$ is not included in this form, when we determine the moduli of periodicity of the real part of this function in such a way that they do not satisfy these conditional equations, which is always possible, following the above reasoning.

Functions ω *which are infinite of the first order in one point of the surface* T *(integral of the second sort)*

Now let us suppose that ω becomes infinite in a single point ε of the surface T, and that for this point all the coefficients in ϕ, with the exception of B, are zero. Such a function is thus determined, to / by ((à)) a additive nearby constant, by the magnitude B and by the real parts of these moduli of periodicity. If we designate by $t^0(\varepsilon)$ an arbitrary function of this nature, in the expression

$$t(\varepsilon) = \beta\, t^0(\varepsilon) + \alpha_1 w_1 + \alpha_2 w_2 + \ldots + \alpha_p w_p + \text{const.},$$

when the constants $\beta, \alpha_1, \alpha_2, \ldots, \alpha_p$ can always be determined in such a way that for this expression the magnitude B and the real parts of the moduli of periodicity take arbitrary given values. This expression represents, therefore, any such function.

Functions ω *which are logarithmically infinite in two points of the surface T (integrals of the third sort)*

Let us consider in the third place the case where the function ω becomes infinite only logarithmically; this must occur, when the sum of magnitudes A is equal to zero, in at least two points of the surface T, ε_1 and ε_2, and it must be the case that $A_2 = -A_1$. If we designate any one of the functions where this is the case, the two last magnitudes being equal to 1, by $\varpi^0(\varepsilon_1,\varepsilon_2)$, all the other functions, by virtue of the conclusions analogous to those employed above, are included in the form

$$\varpi(\varepsilon_1,\varepsilon_2) = \varpi^0(\varepsilon_1,\varepsilon_2) + \alpha_1 w_1 + \alpha_2 w_2 + \ldots + \alpha_p w_p + \text{const.}$$

In the remarks which follow, we will suppose, to simplify the matter, that the points ε are not branch-points and that they are therefore not infinite. One can therefore suppose that $r_\nu = z - z_\nu$, by designating by z_ν the value of z at the point ε_ν. So, when we differentiate $\varpi(\varepsilon_1,\varepsilon_2)$ with respect to z_1 such that the real parts of the moduli of periodicity (or also ρ of the moduli of periodicity) and the value of $\varpi(\varepsilon_1,\varepsilon_2)$ in any point of the surface T remain constant, we obtain a function $t(\varepsilon_1)$ which is discontinuous in ε_1, since it is $1/(z - z_1)$.

Reciprocally, when $t(\varepsilon_1)$ designates such a function, the integral $\int_{z_2}^{z_3} t(\varepsilon_1)\partial z_1$ taken along any line, leading from ε_2 to ε_3, on the surface T, is equal to a function $\varpi(\varepsilon_2,\varepsilon_3)$. In a completely similar manner, we obtain, by n successive differentiations with respect to z_1 of such a function $t(\varepsilon_1)$, functions ω which are discontinuous at the point ε_1, since it is $n!(z - z_1)^{-n-1}$, and which everywhere else remains finite.

For the positions of points ε which we have excluded, these theorems demand a slight modification.

Now, it is evident that one can determine a linear expression with constant coefficients formed by functions ω, from functions ϖ and their derivatives taken with respect to values of discontinuity, and expression such that in the interior of T' it admits of arbitrary given discontinuities of the same form as those of ω, and such that the real parts of the moduli of periodicity take arbitrary given values. One can, consequently, represent any function ω by such an expression.

§ V.

The general expression of a function w, which becomes infinitely great of the first sort in m points $\varepsilon_1, \varepsilon_2, ..., \varepsilon_m$ of the surface T is, following the preceding

$$s = \beta_1 t_1 + \beta_2 t_2 + \dots + \beta_m t_m + \alpha_1 w_1 + \alpha_2 w_2 + \dots + \alpha_p w_p + \text{const.},$$

where t_ν is an arbitrary function $t(\varepsilon_\nu)$ and where the magnitudes α and β are constants. When, among the m points ε, a number ρ of them reunite at the same point η of the surface T, the ρ functions t corresponding to these points must be replaced by a function $t(\eta)$ and be the ρ–1 first derivatives of this function, taken with respect to its value of discontinuity (§II).

The $2p$ moduli of periodicity of this function s are linear homogeneous functions of $p+m$ values α and β. When $m \geqslant p + 1$, among the magnitudes α and β, $2p$ among them can thus be determined as linear homogeneous functions of those which remain, such that the moduli of periodicity are all zero. The function therefore encloses still $m-p+1$ arbitrary constants of which it is a linear homogeneous function, and it can be regarded as a linear expression of $m-p$ functions, among which each one becomes infinite of the first order only for $p+1$ values.

When $m=p+1$, the relationships of the $2p+1$ magnitudes α and β are completely determined for the position of each of the $p+1$ points ε. But, for particular positions of these points, some among the magnitudes β can be equal to zero. Be, for example, $m-\mu$ the number of these magnitudes equal to zero, such that the function becomes infinite of the first sort only for μ points. These m points must therefore have a position such that, among the $2p$ conditional equations among the $p+\mu$ remaining values of β and α, $p+1-\mu$ among them are an identical consequence of those which remain. It follows that there are only $2\mu-p-1$ which can be chosen arbitrarily. Beyond this, the function contains still two more arbitrary constants.

Now, let us take up the determination of s in such a way that μ be as small as is possible. When s is μ-fold infinite of the first order, the same is true of any rational function of s of the first degree. We can therefore, in the resolution of the problem, choose one of the μ points arbitrarily. The position of the others must thus be determined such that $p+1-\mu$ of the conditional equations among the magnitudes α and β be a identical consequence of those which remain.

It is necessary, therefore, when the branch-values of the surface T do not satisfy particular conditional equations, that we have

$$P + 1 - \mu \leqslant \mu - 1 \quad \text{or} \quad \mu \geqslant \tfrac{1}{2}p + 1.$$

The number of arbitrary constants which a function s incorporates, which become infinite of the first order only for m points of the surface T and remains continuous elsewhere, is in all cases equal to $2m-p+1$.

One such function is the root of an equation of degree n whose coefficients are integral functions of z of degree m.

Let $s_1, s_2, \ldots, s_n$ be the n values of the function s for the same value of z and let us designate by σ an arbitrary function; then $(\sigma - s_1)(\sigma - s_2)\ldots(\sigma - s_n)$ is a uniform function of z which becomes infinite only in a point of the z plane coinciding with a point ε, and the order of this infinity will be equal to the number of points ε which there coincide.

Indeed, in a point ε which is not a branch-point, *only one* among the factors of the product is infinite of the first order; for a point ε, around which the surface turns on itself μ times, there are μ infinite factors, each of them being infinite of the order $1/\mu$. If we now designate the values z in *these points* ε, where z is not infinite by $\zeta_1, \zeta_2, \ldots, \zeta_\nu$, and the product $(z - \zeta_1)(z - \zeta_2)\ldots(z - \zeta_\nu)$ by a_0, then $a_0(\sigma - s_1)(\sigma - s_2)\ldots(\sigma - s_n)$ is a uniform function of z which is finite for all finite values of z, and which, for $z=\infty$, becomes infinite of order m; it is thus an integral function of z of the m^{th} degree. It is equally an integral function of σ of the n^{th} degree which disappears for $\sigma=s$. Let us designate it by F, and let us designate, as we will in the following, by $F\left(\overset{n}{\sigma}, \overset{m}{z}\right)$ *an integral function of* F *of degree n in* σ *and degree m in z*; thus s is a root of an equation $F\left(\overset{n}{\sigma}, \overset{m}{z}\right)=0$.

The function F is a power of an irreducible function, which is to say it can not be decomposed into a product of integral functions of σ and of z. Indeed, any rational integral factor of $F(\sigma, z)$ since it must disappear for certain among the roots $s_1, s_2, \ldots s_n$, represents for σ=s a function of z which must disappear in a portion of the surface T and which, it follows, the surface being connected, must be zero all along this surface. But two irreducible factors of $F(\sigma, z)$ can disappear simultaneously for a finite number of pairs of values, only if one of them can not be obtained by multiplying the other by a constant. Consequently, F is necessarily a power of an irreducible function.

When the exponent ν of this power is > 1, the mode-of-branching of the function s is not represented by the surface T, but by a surface τ covering everywhere n/ν times the z plane, and covering itself everywhere ν times on the surface T. Then s can truly be considered as a branching function just as the surface T is, but it cannot be reciprocally claimed that T is branched in the way that s is.

Functions such as s, discontinuous only in certain points of T, is also representable as $d\omega/dz$. Indeed, this function takes the same value on the two sides of cross-cuts and lines l, since the difference between the two values that ω takes on these cuts is constants along the lines. It can be infinite only at points where ω is infinite and in branch-points of the surface, and everywhere else it is continuous, since the derivative of a uniform finite function is also uniform and finite.

All functions ω are therefore algebraic functions of z, branched like surface T, or are integrals of such functions. This system of functions is determined when the surface T is given, and depends only on the position of its branch-points.

§ VI.

Suppose now that we give the irreducible equation

$$F\left(\overset{n}{s},\overset{m}{z}\right)=0$$

and that we try to determine the mode of branching of the function s, or of the surface T which represents it. When, for a value β of z, μ branches of the function come together ((re-attach)) in such a way that one of these branches, after μ circuits around β, is reproduced [comes back to its original], these μ branches of the function (as is easy to demonstrate following Cauchy or by the aid of a Fourier series) can be represented by a series of increasing rational powers of $z - \beta$, where the exponents have for their smallest common denominator μ, and vice versa.

I will name a point of the surface T, where only two branches of the function come together, in such a way that around this point the first branch extends into the second and the second into the first, a *simple branch-point*.

A point of a surface around which branches turn on themselves (μ + 1) times, can therefore be regarded as being μ coinciding simple branch points (or, if not coinciding, infinitely close).

To demonstrate this, let there be, on a portion of the z plane containing this point, $s_1, s_2, \ldots s_{\mu+1}$ uniform branches of the function s, and let $a_1, a_2, \ldots a_\mu$ be simple branch-points situated on the boundary of this portion and following each other in the positive direction of the boundary. Let us suppose that a closed curve drawn in the positive direction around a_1 permutes s_1 and s_2, that a closed curve drawn in the same direction around a_2 permutes s_1 and s_3, …, such that a closed curve drawn in the same direction around a_μ permutes s_1 and $s_{\mu+1}$. Then, after a positive circuit around a domain containing all these points (and only these, no other branch-points included)

$$s_1, \quad s_2, \quad \ldots, \quad s_\mu, \quad s_{\mu+1}$$

are replaced by

$$s_2, \quad s_3, \quad \ldots, \quad s_{\mu+1}, \quad s_1,$$

and, consequently, as they coincide, a branch-point of order μ is created.

The properties of functions ω depend essentially on the order of connectedness of the surface T. To find them, let us first determine the number of simple branch-points of the function s.

At a branch-point, the branches of the function which join together have the same value, and, consequently, two or more roots of the equation

$$F(s) = a_0 s^n + a_1 s^{n-1} + \ldots + a_n = 0$$

become equal. This can only occur when

$$F'(s) = a_0 n s^{n-1} + a_1(n-1)s^{n-2} + \ldots + a_{n-1}$$

or the uniform function of z, $F'(s_1), F'(s_2) \ldots F'(s_n)$, disappears. This function only becomes infinite for finite values of z, when $s = \infty$, and consequently when $a_0 = 0$, and it must be multiplied by a_0^{n-2} to remain finite. It is therefore a uniform function of z, which remains finite for any finite value of z, and which, for $z = \infty$, becomes infinite of order $2m(n\text{-}1)$ and is, consequently, an integral function of degree $2m(n\text{-}1)$. The values of z, for which $F'(s)$ and $F(s)$ disappear simultaneously, are therefore roots of the equation, of degree $2m(n\text{-}1)$,

$$Q(z) = a_0^{n-2} \prod_i F'(s_i) = 0,$$

or yet, since $F'(s_i) = a_0 \prod_{i'} (s_i - s_{i'}),\ (i \gtrless i')$, of the equation

$$Q(z) = a_0^{2(n-1)} \prod_{i,i'} F'(s_i - s_{i'}) = 0, (i \gtrless i'),$$

which one can form by eliminating s in

$$F'(s) = 0 \text{ and } F(s) = 0.$$

If $F(s, z) = 0$ for $s = \alpha$, $z = \beta$, we have

$$\begin{aligned} F(s,z) &= \frac{\partial F}{\partial s}(s-\alpha) + \frac{\partial F}{\partial z}(z-\beta) \\ &\quad + \tfrac{1}{2}\left\{\frac{\partial^2 F}{\partial s^2}(s-\alpha)^2 + 2\frac{\partial^2 F}{\partial s\,\partial z}(s-\alpha)(z-\beta) + \frac{\partial^2 F}{\partial z^2}(z-\beta)^2\right\} \\ &\quad + \cdots, \\ F'(s) &= \frac{\partial F}{\partial s} + \frac{\partial^2 F}{\partial s^2}(s-\alpha) + \frac{\partial^2 F}{\partial s\,\partial z}(z-\beta) + \cdots \end{aligned}$$

Consequently, if, for $s = \alpha$, $z = \beta$, we have

$$\frac{\partial F}{\partial s} = 0,$$

and, if then $\frac{\partial F}{\partial z}, \frac{\partial^2 F}{\partial s^2}$ do not disappear, $s - \alpha$ is infinitely small, since it is $(z-\beta)^{1/2}$, and Q(z) then has $(z - \beta)$ as a factor. In the case where $\frac{\partial F}{\partial z}$ and $\frac{\partial^2 F}{\partial s^2}$ never disappear as long as F and $\frac{\partial F}{\partial s}$ are simultaneously zero, then to each linear factor of Q(z) there corresponds *one* simple branch-point and the number of these points is, consequently, $=2m(n-1)$.

The position of the branch-points depends on the coefficients of the powers of z in the functions a, and varies with them in a continuous manner.

When these coefficients take values such that two simple branch-points, belong to the same pair of branches, coincide, these points destroy each other and two roots of F(s) = 0 become equal to teach other, without a branch-point coming into existence.

If, around each of these points, s_1 extends into s_2, and s_2 into s_1, then, by the effect of a circuit around a portion of the z plane containing these two points, s_1 changes into s_1 and s_2 into s_2, and then at the moment of reunion of two points, the two branches will be uniform. Consequently, their derivative $\frac{ds}{dz}$ is then uniform and finite, and, consequently, we will have

$$\frac{\partial F}{\partial z} = -\frac{ds}{dz}\frac{\partial F}{\partial s} = 0.$$

If $F = \frac{\partial F}{\partial s} = \frac{\partial F}{\partial z} = 0$ for $s = \alpha$, $z = \beta$, then the three terms following the development of F(s, z) will give two values for

$$\frac{s-\alpha}{z-\beta} = \frac{ds}{dz}, \ (s = \alpha, z = \beta).$$

If these values are unequal and finite, the two branches of the function s to which they belong, cannot neither reunite(()) there, nor, consequently, can they come-together-as-a-branch ((ramify)) there. Then $\frac{\partial F}{\partial s}$ becomes infinitely small for the two branches, since it is $z - \beta$, and Q(z) will have $(z - \beta)^2$ as a factor; consequently only two simple branch-points coincide.

In each case where, for $z = \beta$, several roots of the equation F(s) = 0 become equal to α, to distinguish how many simple branch-points coincide for ($s = \alpha$, $z = \beta$), and how many among them are destroyed, one must develop these roots (following the procedure of Lagrange[(1)]), following the increasing powers of $z - \beta$, up to where these developments become altogether different, and we will thus obtain all the branchings which still effectively exist. We must afterwards determine of what order of infinite smallness F'(s) is for each of these roots, in order to determine the number of corresponding linear factors of Q(z), which is to say the number of simple branch-points coinciding for $s = \alpha$, $z = \beta$.

If we designate by ϱ [ρ from now on unless ambiguous], the number indicating how many times the surface T turns on itself around the point (s, z), F'(s) will be at point (z) infinitely small of the first order as many times as there are found coinciding simple branch-points, $dz^{1-1/\rho}$ will so be as many times as there are found effective [active] simple branch-points, and, it follows, F'(s)$dz^{1/\rho-1}$ will so be as many times as among the branch-points, there are those which are destroyed.

Let us designate by w the number of simple effective branch-points, and by $2r$ the number of those which are destroyed, we have

$$w + 2r = 2(n-1)m.$$

If we suppose that the branch-points coincide solely in pairs and thereby destroy each other, we will have, for r pairs of values ($s = \gamma_\rho$, $z = \delta_\rho$),

$$F = \frac{\partial F}{\partial s} = \frac{\partial F}{\partial z} = 0$$

and

$$\frac{\partial^2 F}{\partial z^2}\frac{\partial^2 F}{\partial s^2} - \frac{\partial^2 F}{\partial s \partial z} \gtrless 0,$$

and, for w pairs of s and z, we will have

$$F = 0, \qquad \frac{\partial F}{\partial s} = 0$$

and

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(1) *Nouvelle méthode pour résoudre les equations littérales par le moyen des series*. (Mémoires de l'Académie de Berlin, XXIV; 1780. Œuvres de Lagrange, v. III, p.5) – WEBER. [Note added to French translation – JAR]

$$\frac{\partial F}{\partial z} \gtrless 0, \qquad \frac{\partial^2 F}{\partial s^2} \gtrless 0.$$

We will hold ourselves, in general, to the consideration of this last case, for the results can be easily extended to others, considered as limiting cases; and we can do this as easily as we have laid down the theory of these functions on principles independent of their form of expression and which are not subjected to any exception.

§ VII.

Now, relative to a simply connected surface covering a finite portion of the z plane, the following relation holds among the number of simple branch-points and the number of circuits formed by the run of its enclosing curve: this latter number is one greater than the first; by the aid of this relation we can derive another, relative to a multiply connected surface, among these numbers and the number of cross-cuts which decompose this surface into a simply connected surface.

This relation, fundamentally independent of all metric relation and which relate to *analysis situs*, can be deduced for the surface T as follows.

By virtue of Dirichlet's Principle, on the simply connected surface T', the function of z, log ζ, can be determined in such a way that ζ in any point in the interior of this surface is infinitely small of the second order, and that log ζ, along any line which does not cross itself, and going from this point to reach again the boundary, be larger by $-2\pi i$ on the positive side of this line than on the negative side, and be always continuous and purely imaginary along the boundary of T'. Then the function ζ takes one time each value whose modulus is < 1. The totality of its values will be thus represented by a surface covering one time a circle on the ζ plane. To each point of T' there therefore corresponds a point of the circle, and vice-versa. Consequently, in any point of the surface where $z=z'$, $\zeta=\zeta'$, the function $\zeta-\zeta'$ is infinitely small of the first order and, consequently, in this point, when the surface T' turns on itself $\mu + 1$ times around this point, for a finite z',

$$(\mu+1)\frac{z-z'}{(\zeta-\zeta')^{\mu+1}} = \frac{dz}{d\zeta(\zeta-\zeta')^{\mu}},$$

remains finite, but when z' is infinite, it is

$$(\mu+1)\frac{z^{-1}}{(\zeta-\zeta')^{\mu+1}} = -\frac{\partial z}{zz\partial\zeta(\zeta-\zeta')^{\mu}}$$

which remains finite. The integral $\int d\log\frac{dz}{d\zeta}$, taken in the positive direction around the entire boundary of the circle, is equal to the sum of the integrals taken around the points where $\frac{dz}{d\zeta}$ is infinite or zero, and, it follows, is equal to $2\pi i(\mathrm{w}-2n)$.

Let us designate by s a portion of the boundary of T', going from one and the same point determined by this boundary, to a variable point on the same boundary, and by σ the corresponding portion on the boundary of the circle; we will have

$$\log\frac{\partial z}{\partial\zeta} = \log\frac{\partial z}{\partial s} + \log\frac{\partial s}{\partial\sigma} - \log\frac{\partial\zeta}{\partial\sigma};$$

for the integrals taken along the whole boundary we will have

$$\int \partial \log \frac{\partial z}{\partial s} = (2p-1)2\pi i,$$

$$\int \partial \log \frac{\partial z}{\partial \sigma} = 0,$$

$$-\int \partial \log \frac{\partial \zeta}{\partial \sigma} = -2\pi i,$$

and, thus,

$$\int \partial \log \frac{\partial z}{\partial \zeta} = (2p-2)2\pi i.$$

We have therefore w – 2n = 2(p – 1), from which

$$w = 2((n-1)m - r),$$

thus,

$$p = (n-1)(m-1) - r.$$ [2]

§ VIII.

The general expression of s', functions of z, branched as is the surface T, which, for m' arbitrarily given points of T, become infinite of the first order and which everywhere else remain continuous, contains, considering what has come before, $m' - p + 1$ arbitrary constants and is a linear function of them (§ V). It follows that, if it is possible, as will soon be demonstrated, to form rational expressions of s and z, which become infinite of the first order for m' pairs of arbitrarily given values of s and z, satisfying the equation F = 0, and which are linear functions of $m' - p + 1$ arbitrary constants, then any function s' can be represented by these expressions.

In order that the quotient of two integral functions $\chi(s, z)$ and $\psi(s, z)$ can take, for $s = \infty$ and $z = \infty$, arbitrary finite values, the two functions must be of the same degree.

Let us then designate the expression to represent s' by $\dfrac{\psi(\overset{\nu}{s}, \overset{\mu}{z})}{\chi(\overset{\nu}{s}, \overset{\mu}{z})}$, and also let $\nu \geqq n - 1$, $\mu \geqq m - 1$. When two branches of the function s become equal without one extending into the other, and when we have, consequently, in two distinct points of the surface T, $z = \gamma$, $s = \delta$, s' in general will take different values in these two points; in order to have everywhere $\psi - s'\chi = 0$, we must therefore have, for two different values of s', $\psi(\gamma,\delta) - s'\chi(\gamma,\delta) = 0$, and, consequently, $\chi(\gamma,\delta) = 0$ and $\psi(\gamma,\delta) = 0$.

Thus, the functions χ and ψ disappear for the r pairs of values ($s = \gamma_\rho$, $z = \delta_\rho$)[4].

The function χ disappears, for a value of z, for which the uniform-in-z function, finite for finite z,

$$K(z) = a_0^{\nu}\,\chi(s_1)\,\chi(s_2)\,\dots\,\chi(s_n) = 0.$$

[4] Here, as has been previously indicated, we will not occupy ourselves with the case where the branch-points of the function s coincide only by pairs in destroying themselves. In general, the functions χ and ψ, in a point of T, where there coincide, in keeping with § VI, the branch-points which destroy each other when T turns ρ times around this point, must become infinitely small as does $F'(s)dz^{1/\rho-1}$, in order that in the development of the series of whole powers of $(\delta z)^{1/\rho}$ of the function to be represented, the first terms can take arbitrary values. – (RIEMANN.)

This function, for infinite z, becomes infinite of order $m\nu + n\mu$; it is, thus, an integral function of degree $m\nu + n\mu$.

Thus, for the pairs of values (γ,δ), two factors of the product $\prod_i \chi(s_i)$ become infinitely small of the first order, and that, therefore, $K(z)$ becomes infinitely small of the second order, then χ will additionally be infinitely small of the first order for

$$i = m\nu + n\mu - 2r$$

pairs of values of s and z, which is to say of points on T.

If $\nu > n - 1$, $\mu > m - 1$, then the value of the function χ remains unchanged when we replace $\chi(\overset{\nu}{s}, \overset{\mu}{z})$ by

$$\chi(\overset{\nu}{s}, \overset{\mu}{z}) + \rho(\overset{\nu-n}{s}, \overset{\mu-m}{z}) F(\overset{n}{s}, \overset{m}{z}),$$

ρ being arbitrary; thus it follows that among the coefficients of this expression there are

$$(\nu - n + 1)(\mu - m + 1)$$

which can be taken arbitrarily. Now, if among the

$$(\mu + 1)(\nu + 1) - (\nu - n + 1)(\mu - m + 1)$$

which remain, we can determine r of them as linear functions of the others, such that χ disappear for the r pairs of values (γ,δ), then function χ still contains

$$\varepsilon = (\mu + 1)(\nu + 1) - (\nu - n + 1)(\mu - m + 1) - r$$
$$= n\mu + m\nu - (n - 1)(m - 1) - r + 1$$

arbitrary constants. We have, therefore

$$i - \varepsilon = (n - 1)(m - 1) - r - 1 = p - 1.$$

Now, if we choose μ and ν such that we have $\varepsilon > m'$, we can determine χ such that, for m' pairs of arbitrarily given values, this function becomes infinitely small of the first order, and then, when $m' > p$, we can dispose of ψ such that $^{\psi}/_{\chi}$ remains finite for all the other remaining values.

In effect, ψ is equally a linear homogenous function of ε arbitrary constants, and, consequently, we can, when $\varepsilon - i + m' > 1$, determine $i - m'$ among them as functions of those which remain, such that ψ disappears equally for the $i - m'$ pairs of values of s and z for which χ is still infinitely small of the first order. The function ψ contains therefore

$$\varepsilon - i + m' = m' - p + 1$$

arbitrary constants, and, thus, $^{\psi}/_{\chi}$ can represent any function s'.

§ IX.

Since the functions $\frac{\partial \omega}{\partial z}$ are algebraic functions of z branching in the same way as is s, they can, by virtue of the proposition which was just demonstrated, be expressed rationally in terms of s and z, and all the functions ω can be represented as integrals of rational functions of s and z.

If we designate by w a function ω which is everywhere finite, $\frac{\partial w}{\partial z}$ will be infinite of the first order in each simple branch-point, but everywhere else it will be continuous, and, for $z = \infty$, it is infinitely small of the second order. Vice versa, the integral of a function which behave thus will remain everywhere finite.

To express this function $\frac{\partial w}{\partial z}$ as a quotient of two integral functions of *s* and *z*, we must (following section § VII) make the denominator a function which vanishes at the branch-points and for the *r* pairs of values (γ,δ). This condition can be fulfilled in the simplest fashion by taking a function which becomes zero for only these values and

$$\frac{\partial F}{\partial s} = a_0 n s^{n-1} + a_1 \overline{n-1} s^{n-2} + \ldots + a_{n-1}$$

is such a function.

This function, for an infinite *s*, becomes infinite of order $(n-2)$ (since a_0 is therefore infinitely small of the first order), and, for infinite *z*, it must become infinite of order *m*.

In addition, $\frac{\partial w}{\partial z}$ must remain finite, save for the branch-points, and for infinite *z*, must remain finitely small of the second order. The numerator must be, consequently, an integral function $F(\overset{n-2}{s}, \overset{m-2}{z})$, which vanishes for the *r* pairs of values (γ,δ) (see § VI).

We therefore have

$$w = \int \frac{\varphi(\overset{n-2}{s}, \overset{m-2}{z})\,\partial z}{\frac{\partial F}{\partial s}} = -\int \frac{\varphi(\overset{n-2}{s}, \overset{m-2}{z})\,\partial s}{\frac{\partial F}{\partial z}},$$

where $\varphi = 0$ for $s = \gamma_\rho$, $z = \delta_\rho$, with $\rho = 1, 2, \ldots, r$.

The function φ has $(n-1)(m-1)$ constant coefficients, and when *r* among them are determined as linear functions of those which remain such that $\varphi = 0$ for the *r* pairs of values $s = \gamma$, $z = \delta$, then there yet remain $(m-1)(n-1) - r$, which we will call *p* which are arbitrary and φ takes thus the form

$$\alpha_1\varphi_1 + \alpha_2\varphi_2 + \ldots + \alpha_p\varphi_p,$$

where $\varphi_1, \varphi_2, \ldots, \varphi_p$ are particular functions φ, of which none is a function of those which remain, and where $\alpha_1, \alpha_2, \ldots, \alpha_p$ designate arbitrary constants.

For the most general expression of *w*, we obtain, as has been done above [see § IV] in another manner,

$$\alpha_1 w_1 + \alpha_2 w_2 + \ldots + \alpha_p w_p + \text{const.}$$

The functions ω which are not everywhere finite, and, consequently, the integrals of the second and third degree, can, by virtue of the same principles, be expressed rationally in terms of *s* and *z*, but we will not end with this subject, in view of the fact that the general rules of the preceding paragraph have no need of further elaboration, and that, for the consideration of determined forms of these integrals, the theory of ϑ-functions will furnish us the first occasion to return.

§ X.

The function φ will be infinitely small of the first order, beyond for the *r* pairs of values (γ, δ), also for

$$m(n-2) + n(m-2) - 2r$$

which is to say, for 2 (p – 1) pairs of values of s and z, satisfying the equation F = 0. Let now

$$\varphi^{(1)} = \alpha_1^{(1)}\varphi_1 + \alpha_2^{(1)}\varphi_2 + \cdots + \alpha_p^{(1)}\varphi_p$$

and

$$\varphi^{(2)} = \alpha_1^{(2)}\varphi_1 + \alpha_2^{(2)}\varphi_2 + \cdots + \alpha_p^{(2)}\varphi_p$$

be two arbitrary functions φ; we can therefore in the expression $\varphi^{(2)}/\varphi^{(1)}$ first of all determine the denominator, such that it is null for p – 1 pairs of given arbitrary values of s and z satisfying the equation F = 0, and after that the numerator, such that it also disappears for p – 2 among these pairs of values for which $\varphi^{(1)}$ is equal to zero. It is therefore a linear function of two arbitrary constants, and, consequently, it is the general expression of a function which becomes infinite of the first order in only p points of the surface T. A function which becomes infinite for less than p points forms a special case of this function. Consequently, all the functions which become infinite of the first order for fewer than p + 1 points of the surface T can be represented in the form $\varphi^{(2)}/\varphi^{(1)}$ or in the form $dw^{(2)}/dw^{(1)}$, when $w^{(1)}$ and $w^{(2)}$ are two everywhere-finite integrals of rational functions in terms of s and z.

§ XI.

A function z_1 of z, branched in a like manner to T, which becomes infinite of the first order for n_1 points of this surface is, by virtue of the preceding (§ V), a root of an equation of the form

$$G\left(\overset{n}{z_1}, \overset{n_1}{z}\right)=0,$$

and, it follows that it takes each value for the n_1 points of the surface T. [[Unclear sentence]] Consequently, when we imagine each point on T represented by a point on a plane which geometrically represents the value of z_1 at this point, the totality of these points forms a surface T_1 which covers the z_1 plane n_1 times, a surface which is, as we know, a representation, equal in its smallest parts to the surface T. To each point on one of these surfaces there corresponds, then, a *unique* point on the other. Functions ω, i.e. the integrals of the functions of z, branched as is T, are transformed, when in place of z, we introduce z_1 as the independent variable, into functions which have everywhere a *unique* determined value on the surface T_1 and have the same discontinuities as the functions ω at the corresponding points of T, and which, it follows, are integrals of the functions of z_1, branched as T_1 is.

If we designate by s_1 any other arbitrary function of z, branched as T is, which becomes infinite of the first order for m_1 points of T and thus also m_1 points of T_1, then (§V) an equation holds between s_1 and z_1 of the form

$$F_1\left(\overset{n_1}{s_1}, \overset{m_1}{z_1}\right)=0,$$

where F_1 is a power of an entire irreducible function of s_1, z_1, and we can, express all functions of z_1 branched as is T_1 in terms of s_1 and z_1, when this power has unity as its exponent, and, similarly, all rational functions of s and z (§VIII).

The equation $F\left(\overset{n}{s}, \overset{m}{z}\right)$=0 can therefore be transformed into $F_1\left(\overset{n_1}{s_1}, \overset{m_1}{z_1}\right)$=0, and *vice versa*, by the use of a rational transformation.

The domains of the magnitudes (s, z) and (s_1, z_1) have therefore the same degree of connectedness, because at each point of the one corresponds a *unique* point of the other. If we therefore designate by r_1 the number of cases where s_1 and z_1 both take the same value, for two different points of the domain of magnitudes (s_1, z_1), cases for which, it follows, we have simultaneously F_1, $\frac{\partial F_1}{\partial s_1}$, and $\frac{\partial F_1}{\partial z_1}$ = 0, and

$$\frac{\partial^2 F_1}{\partial s_1^2}\frac{\partial^2 F_1}{\partial z_1^2} - \left(\frac{\partial^2 F_1}{\partial s_1 \partial z_1}\right)^2 \neq 0,$$

one must necessarily have

$$(n_1 - 1)(m_1 - 1) - r_1 = p = (n - 1)(m - 1) - r$$

§ XII

We will now consider as all being part of the *same class, all irreducible algebraic equations of two variables, which can be transformed into each other by rational substitutions*; of the sort

$$F(s, z) = 0 \qquad \text{and} \qquad F_1(s_1, z_1) = 0$$

belonging to the same class when s and z can be replaced by rational functions of s_1 and z_1, such that $F(s, z) = 0$ is transformed into $F_1(s_1, z_1) = 0$, s_1 and z_1 both being rational functions of s and z.

Rational functions of s and z, considered as functions of an arbitrary ζ among them, form a system of algebraic functions of the same branching characteristics. [[unclear around zeta]]

In this manner, any equation leads evidently to a class of systems of algebraic functions with the same branchings which, by the introduction of a functin of the system as an independent variable, are transformable into each other, and this in such a way that all the equations of *one* class lead to the same class of systems of algebraic functions; and reciprocally (§XI), any class of such systems leads to *one* class of equations. If the domain of magnitudes (s, z) is ($2p$ + 1)-ply connected and if the function ζ becomes infinitely small of the first order in μ points of this domain, the number of branch-values of functions of ζ, to the same branchings, which are formed by other rational functions of the remaining s and z, is equal to 2(μ + p – 1), and the number of arbitrary constants contained in the function ζ is $2\mu - p + 1$ (§V). These constants can be determined in such a way that $2\mu - p + 1$ branch-values take given values, when these branch-values are functions which are independent among themselves of these constants, and this only in a finite number of manners, since the condition-equations are algebraic. In each class of systems of same-branched and (2p + 1)-ply connected functions, there exist therefore a finite number of systems of μ-valued functions, for which $2\mu - p + 1$ branch-values take given values. On the other hand, when the $2(\mu + p - 1)$ branch-points of a ($2p$ + 1)-ply connected surface, completely covering the ζ plane μ times, are given arbitrarily, there always exists (§III-V) a system of algebraic functions of ζ, of the same branchings as this surface. The $3p - 3$ branch-values remaining in these systems of same-branched μ-valued functions can therefore take any values; consequently, a class of systems of same-branched and $(2p + 1)$-ply connected functions and the class of algebraic equations which belongs to it, depend on $3p - 3$ magnitudes varying in a continuous manner, which will be named the *moduli of the class*.

This determination of the number of moduli of a class of algebraic $(2p + 1)$-ply connected functions, is valid only under the hypothesis that there are $2\mu - p + 1$ branch-points which are mutually independent functions of arbitrary constants which contain the function ζ. This hypothesis is only valid when $p > 1$; the number of moduli is thus equal to $3p - 3$; but, for $p = 1$, there number is 1. The direct study of this number is difficult because of the manner that the arbitrary constants enter into ζ. To determine the number of moduli, we will introduce as an independent variable in a system of $(2p + 1)$-ply connected, same-branched functions, not one of these functions, but rather an everywhere-finite integral of such a function. The values which the function w of z takes on the surface T' will be represented geometrically by a surface covering one or many times a finite portion of the w plane, a surface which we will designate by S and which is a representation (equal in smallest parts) of the surface T'. Since the value of w on the positive side of the ν^{th} cross-cut is greater by the amount of the additive constant $k^{(\nu)}$ than the value that it takes on the negative side, the boundary of S is formed by pairs of parallel curves which are the representation of the same portion of the system of cuts which form the boundary of T', and the difference of the location of the corresponding points on parallel portions of the boundary of S, which are the representation of the ν^{th} cross-cut, will be expressed by the complex value $k^{(\nu)}$. The number of simple branch points of the surface S is $2p - 2$, since dw is infinitely small of the second order in $2p - 2$ points of the surface T. Rational functions of s and z are thus functions of w, which, for each point of S, have a *unique* value changing in a continuous manner where they are not infinite, and which retake the same value on corresponding points on parallel bounding portions. They form thus a system of same-branched, $2p$-ply periodic functions w. Now (by a path analogous to that of § III-V), treating as given (by hypothesis) the $2p - 2$ branch-points and the $2p$ differences of situation of parallel bounding portions of the surface S, we can demonstrate that there always exists a system of same-branched functions, such that this surface, which for corresponding points on the parallel boundary portions, takes the same value, and are consequently, $2p$-ply periodic and which, regarded as functions of each other, form a system of same-branched $(2p + 1)$-ply connected algebraic functions and therefore lead to a class of $(2p + 1)$-ply connected algebraic functions. In effect, by virtue of Dirichlet's Principle, a function of w is determined on the surface S, ((à une constante additive près)), by these conditions: in the interior of S it admits of arbitrarily given discontinuities of the same form as those of w on T', and on the correspoinding points on the parallel bounding portions it will take values which differ from constants with given real parts. We conclude from this, as we have done in a similar manner in § V, the possibility of the existence of functions which become discontinuous only in isolated points of S and which have the same value at corresponding points of parallel bounding sections. If such a function z is first-order infinite at n points of S and is nowhere else discontinuous, it will therefore take every complex value at n points of S; in effect, with a designating an arbitrary constant, the integral $\int d \log (z - a)$, taken around S, is zero, since the integrals taken along the parallel bounding sections destroy themselves, and $z - a$ becomes first-order infinitely small on the surface S as many times as it becomes first-order infinitely large. The values which z takes will therefore be represented by a surface n-times completely covering the z plane, and the other same-branched periodic functions of w form a system of $(2p + 1)$-ply connected algebraic functions of z, which are branched in the same way as is the surface. QED.

Now, given an arbitrary class of $(2p + 1)$-ply connected algebraic functions, we can, in the magnitude which we introduce as an independent variable

$$w = \alpha_1 w_1 + \alpha_2 w_2 + \ldots + \alpha_p w_p + c,$$

determine the values of α, such that p among the $2p$ moduli of periodicity take given values, as does the constant c when $p > 1$, such that one of the $2p - 2$ branch-values of the periodic functions of w takes a given value. In this manner w is completely determined, and thus so are the $3p - 3$ remaining magnitudes on which the mode-of-branching and the periodicity of these functions w depend; and since a class of $(2p + 1)$-ply connected functions corresponds to arbitrary values of these $3p - 3$ magnitudes, such a class depends on $3p - 3$ independent variables.

When $p = 1$, there are no branch-points, and in the expression

$$w = \alpha_1 w_1 + c,$$

the magnitude α can be determined such that *one* of the moduli of periodicity takes a given value, and the other modulus is determined by this one.

The number of moduli of one class is therefore in this case equal to 1.

§ XIII.

Following the preceding principles of transformation (developed in § XI), to transform by means of a rational substitution any given equation $F(s, z) = 0$ into an equation

$$F_1\left(\overset{n_1}{s_1}, \overset{m_1}{z_1}\right) = 0$$

of the same class and of the lowest possible degree, we must first determine for z_1 an expression $r(s, z)$, rational in s and z, such that n_1 be as small as possible, and then also determine for s_1 another rational expression $r'(s, z)$ such that m_1 be as small as possible and such that at the same time the values of s_1 corresponding to any value of z_1, are not distributed in groups of equal value among them, such that $F_1\left(\overset{n_1}{s_1}, \overset{m_1}{z_1}\right)$ cannot be a power greater than one of an irreducible function.

When the domain of magnitudes (s, z) is $(2p + 1)$-ply connected, the smallest value which n_1 can take is, generally, $\geq p/2 + 1$ (§ V), and the number of cases where s_1 and z_1 both take the same value for two different points of the domain of magnitudes is equal to

$$(n_1 - 1)(m_1 - 1) - p.$$

In a class of algebraic equations of two variable magnitudes, when the moduli of the class are not conditional on the verifying of equations of particular conditions ((POOR translation)), the equations of minimal degree consequently have the following form:

for $p = 1$, $\mathrm{F}\left(\overset{2}{s}, \overset{2}{z}\right) = 0$, $r = 0$

$p = 2$, $\mathrm{F}\left(\overset{2}{s}, \overset{3}{z}\right) = 0$, $r = 0$

$p > 2$ { $p = 2\mu - 3$, $\mathrm{F}\left(\overset{\mu}{s}, \overset{\mu}{z}\right) = 0$, $r = (\mu - 2)^2$

{ $p = 2\mu - 2$, $\mathrm{F}\left(\overset{\mu}{s}, \overset{\mu}{z}\right) = 0$, $r = (\mu - 1)(\mu - 3)$.

Among the coefficients of the powers of *s* and *z* in the integral functions F, we must determine *r* to be linear homogeneous functions of those which remain such that $\frac{\partial F}{\partial s}$ and $\frac{\partial F}{\partial z}$ disappear simultaneously for *r* pairs of values satisfying the equation F = 0. The rational functions of *s* and *z*, considered as functions of one among them, represent then all the systems of (2*p* + 1)-ply connected algebraic functions.

§ XIV.

I will now make use, following Jacobi[(1)] (*Crelle's Journal*, vol. 9, No. 32, § VIII), of the addition theorem of Abel for the integration of a system of differential equations. On this point, I will confine myself to what will later be necessary in the course of this report.

When in *w,* an everywhere-finite integral of a rational function of *s* and *z*, we introduce as an independent variable, ζ, which is a rational function of *s* and *z*, which, for *m* pairs of values of *s* and *z*, becomes first-order infinite, thus $\frac{\partial w}{\partial z}$ is an *m*-valued function of ζ. If we designate the *m* values of *w* for the same ζ by $w^{(1)}, w^{(2)}, \ldots, w^{(m)}$, then

$$\frac{\partial w^{(1)}}{\partial \zeta} + \frac{\partial w^{(2)}}{\partial \zeta} + \ldots + \frac{\partial w^{(m)}}{\partial \zeta}$$

is a uniform function of ζ whose integral remains everywhere finite, and consequently, the integral

$$\int \partial(w^{(1)} + w^{(2)} + \ldots + w^{(m)})$$

is also everywhere uniform and finite, and, it follows, is also equal to a constant. In a analogous manner, we find that, $\omega^{(1)}, \omega^{(2)}, \ldots, \omega^{(m)}$ designating the values corresponding to the same ζ, of an arbitrary integral ω of a rational function of *s* and *z*, the integral

$$\int \partial(\omega^{(1)} + \omega^{(2)} + \ldots + \omega^{(m)})$$

is determined by an additive constant ((près)) by means of the discontinuities of ω, and that under the form of a sum of a rational function of ζ and of logarithms of rational functions of ζ affected by constant coefficients.

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(1) *Works*. Vol. II, p. 15. – (W. and D.) [note added by French translator?]

By the aid of this theorem, we can integrate (as will be soon demonstrated) in a general or complete manner, the p simultaneous differential equations among the $p + 1$ pairs of values (s_1, z_1), (s_2, z_2), …, (s_{p+1}, z_{p+1}) of s and z satisfying the equation $F(s, z) = 0$,

$$\frac{\varphi_\pi(s_1, z_1)\,\partial z_1}{\frac{\partial F(s_1, z_1)}{\partial s_1}} + \frac{\varphi_\pi(s_2, z_2)\,\partial z_2}{\frac{\partial F(s_2, z_2)}{\partial s_2}} + \frac{\varphi_\pi(s_{p+1}, z_{p+1})\,\partial z_{p+1}}{\frac{\partial F(s_{p+1}, z_{p+1})}{\partial s_{p+1}}} = 0$$

where $\pi = 1, 2, \ldots, p$.

By the aid of these differential equations, p among the magnitudes (s_μ, z_μ), are completely determined as functions of one of the remaining pairs, when, for an arbitrary value of this last, the values of the others are given.

Consequently, when we determine these $p + 1$ pairs of magnitudes as functions of a unique variable ζ, such that, for the same value zero of this magnitude, they take the *arbitrarily given* initial values (s_1^0, z_1^0), (s_2^0, z_2^0), …, (s_{p+1}^0, z_{p+1}^0), and satisfy the differential equations, we thus have by the same token integrated the differential equations in a general manner. Now the magnitude $1/\zeta$ can always be determined as a uniform function, and one that is rational in (s, z), such that it is only infinite, and only first-order infinite for all of, or a part of the $(p + 1)$ pairs of values (s_μ^1, z_μ^1), since, in the expression

$$\sum_{\mu=1}^{\mu=p+1} \beta_\mu t(s_\mu^0, z_\mu^0) + \sum_{\mu=1}^{\mu=p} \alpha_\mu w_\mu + \text{const.}$$

the relationships among the magnitudes α and β can always be determined so that the moduli of periodicity both be equal to zero. So, when none of the β are equal to zero, the $(p + 1)$ branches of the functions s and z of ζ, that is, same-branched $(2p + 1)$-valued functions of ζ, (s_1, z_1), (s_2, z_2), …, (s_{p+1}, z_{p+1}), which, for $\zeta = 0$, take the values (s_1^0, z_1^0), (s_2^0, z_2^0), …, (s_{p+1}^0, z_{p+1}^0), satisfy the differential equations we are trying to resolve. But, when among the magnitudes β, certain among them are equal to zero, for example the $p + 1 - m$ last ones, then the differential equations we seek to solve are satisfied by the m branches of the m-valued functions of ζ, (s_1, z_1), (s_2, z_2), …, (s_m, z_m) which, for $\zeta = 0$, are equal to (s_1^0, z_1^0), (s_2^0, z_2^0), …, (s_m^0, z_m^0), and by the constant values of magnitudes s_{m+1}, z_{m+1}, …; s_{p+1}, z_{p+1}, that is to say consequently, by their initial values s_{m+1}^0, z_{m+1}^0, …; s_{p+1}^0, z_{p+1}^0. In this last case, among the p linear homogeneous equations

$$\sum_{\mu=1}^{\mu=m} \frac{\varphi_\pi(s_\mu, z_\mu)\,\partial z_\mu}{\frac{\partial F(s_\mu, z_\mu)}{\partial s_\mu}} = 0 \qquad (\pi = 1, 2, \ldots, p)$$

among the magnitudes

$$\frac{\partial z_\mu}{\frac{\partial F(s_\mu, z_\mu)}{\partial s_\mu}}$$

$p + 1 - m$ among them are a consequence of the others. This gives therefore $p + 1 - m$ conditional equations which (in order for this case to present itself) must hold among the functions (s_1, z_1), (s_2, z_2), …, (s_m, z_m) and consequently also among their initial values (s_1^0, z_1^0), (s_2^0, z_2^0), …, (s_m^0, z_m^0); it follows that as we have already seen in §V, among these, only $2m - p - 1$ can be given arbitrarily.

§ XV.

Let us now designate by w_π the integral

$$\int \frac{\varphi_\pi(s,z)\,\partial z}{\dfrac{\partial F(s,z)}{\partial s}} + \text{const.},$$

taken on the interior of surface T', and let us designate by $k_\pi^{(\nu)}$ the modulus of periodicity of w_π relative to the ν^{th} cross-cut, doing this so that the functions $w_1, w_2, \ldots, w_p$ of pairs of magnitudes (s, z) go from the negative to the positive side of the ν^{th} cross-cut while undergoing an increase of $k_1^{(\nu)}$, $k_2^{(\nu)}$, …, $k_p^{(\nu)}$. To be brief, one can say that a system of p magnitudes $(b_1, b_2, \ldots, b_p)$ is *congruent to another* $(a_1, a_2, \ldots, a_p)$ *relative to* 2*p systems of conjugate moduli*, when it can be derived from this other system by the simultaneous addition of conjugate moduli ((zusammengehöriger – together-fitness?)) to all the respective elements of the system.

This, if the modulus of the π^{th} magnitude in the ν^{th} system is $k_\pi^{(\nu)}$, one will say that

$$(b_1, b_2, \ldots, b_p) = (a_1, a_2, \ldots, a_p)$$

when

$$b_\pi = a_\pi + \sum_{\nu=1}^{\nu=2p} m_\nu k_\pi^{(\nu)}$$

with $\pi = 1, 2, \ldots, p$, and $m_1, m_2, \ldots, m_{2p}$ being integers.

Since p arbitrary magnitudes, $a_1, a_2, \ldots, a_p$ can always and in a unique manner be expressed in the form $a_\pi = \sum_{\nu=1}^{\nu=2p} \xi_\nu k_\pi^{(\nu)}$ such that the 2p magnitudes ξ be real; and since by varying these ξ to different integers we obtain all the systems congruent to this system of magnitudes $a_1, a_2, \ldots, a_p$ and only these systems, we will therefore obtain one and only one system of each series of congruent systems, when, in these expressions, we continuously vary each magnitude ξ, in causing it to take successively all the values from an arbitrary value up to a value that is 1 greater than one of these two limits comprised in the interval.

This posed, from the preceding differential equations, or from the p equations

$$\sum_{\mu=1}^{\mu=p+1} dw_\pi^{(\mu)} = 0 \quad (\pi = 1, 2, \ldots, p)$$

we obtain, by integration

$$\left(\sum w_1^{(\mu)}, \sum w_2^{(\mu)}, \ldots, \sum w_p^{(\mu)}\right) \equiv (c_1, c_2, \ldots, c_p),$$

where the $c_1, c_2, \ldots, c_p$ are constant magnitudes which depend on the values (s^0, z^0).

§ XVI.

If we express ζ as the quotient $\frac{\chi}{\psi}$ of two integral functions of s and z, the pairs of magnitudes $(s_1, z_1), \ldots, (s_m, z_m)$ are the common roots of the equations

$$F=0 \text{ and } \frac{\chi}{\psi}=\zeta.$$

Since the integral function

$$\chi - \zeta\psi = f(s, z)$$

disappears for all the pairs of values for which χ and ψ vanish simultaneously for any ζ, then the pairs of magnitudes $(s_1, z_1), \ldots, (s_m, z_m)$ can also be defined as common roots of the equation $F = 0$ and of an equation $f(s, z) = 0$, whose coefficients vary in such a way that all the other common roots remain constant. When

$$m < p + 1,$$

ζ can be represented (§X) in the form

$$\frac{\varphi^{(1)}}{\varphi^{(2)}}$$

and f can be represented in the form

$$\varphi^{(1)} - \zeta\varphi^{(2)} = \varphi^{(3)}.$$

The most general values of the pairs of functions $(s_1, z_1), \ldots, (s_p, z_p)$ satisfying the p equations

$$\sum_{\mu=1}^{\mu=p} dw_\pi^{(\mu)} = 0 \quad (\pi = 1, 2, \ldots, p)$$

will therefore be formed by p common roots of the equations $F = 0$ and $\varphi = 0$, which vary in such a way that the other common roots remain constant. From which we easily conclude a proposition which will be necessary for what follows, a proposition stating that the problem of the determination of $(p - 1)$ among the $(2p - 2)$ pairs of magnitudes

$$(s_1, z_1), \ldots, (s_{2p-2}, z_{2p-2})$$

as functions of the $(p - 1)$ which remain, and that in a way that the p equations

$$\sum_{\mu=1}^{\mu=p} dw_\pi^{(\mu)} = 0 \quad (\pi = 1, 2, \ldots, p)$$

be satisfied, is solved in a completely general manner when for these $2p - 2$ pairs of values, we choose the common roots of the two equations $F = 0$, $\varphi = 0$, which are different from the r roots $s = \gamma_\rho$, $z = \delta_\rho$ (§VI), that is, the $2p - 2$ pairs of values for which dw becomes second-order infinitely small; so that the problem has a *unique* solution. Similar pairs of magnitudes are called *associated by the extremes of the equation* $\varphi = 0$. From the equations

$$\sum_{\mu=1}^{\mu=p} dw_\pi^{(\mu)} = 0 \quad (\pi = 1, 2, \ldots, p)$$

comes consequently the system

$$\left(\sum_1^{2p-2} w_1^{(\mu)}, \sum_1^{2p-2} w_2^{(\mu)}, \ldots, \sum_1^{2p-2} w_p^{(\mu)}\right)$$

the sums being taken over pairs of associated magnitudes, is congruent to a system $(c_1, c_2, \ldots, c_p)$ of constant magnitudes, where c_π depends only on the additive constant in the function w_π, that is on the initial value of the integral expressing this function.

[2] If we pose

$$dz = ds\, e^{i\phi},$$

ϕ will be the angle between the direction of the element ds of the boundary curve, and the x-axis, and, it follows, that the integral

$$\frac{1}{2\pi i}\int d\log\frac{dz}{ds} = \frac{1}{2\pi}\int d\varphi$$

is equal to the number of circuits described by the direction of the boundary line when we travel it in the positive direction. Then each cross-cut will be crossed two times, in opposite directions, such that these portions of the path destroy each other and that there remain only the parts relative to the $2p - 1$ network-knots of cross cuts (§ III), each contributing to the value 2π. We thus obtain the relation

$$\mathrm{w} - 2n = 2(p - 1),$$

a formula for which the proposition which begins § VII is the translation.

A demonstration of this theorem without the use of Dirichlet's Principle, and which is completely independent of metric relations, has been given by C. Neumann (*Vorlesungen über Riemann's Theorie der Abelschen Integrale*, Chap. VII, § 8, 2nd Edition, Leipzig, Teubner; 1884).