Theory of Abelian Functions
Part II

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Crelle's Journal – v. 54 1857

SECTION II.

§ XVII.

For the subsequent study of the integrals of $(2p + 1)$-ply connected algebraic functions, it will be of great benefit to consider a p-ply infinite series ϑ, a p-ply infinite series where the logarithm of the general term is an integral function of the second degree of the index. In this function, for a term with indices $m_1, m_2, \ldots, m_p$, we will designate the coefficient of the square m_μ^2 by $a_{\mu,\mu}$, that of the double product $m_\mu m_{\mu'}$ by $a_{\mu,\mu'} = a_{\mu',\mu}$, that of double the magnitude m_μ by v_μ, and the constant term will be taken as equal to zero. The sum of the series, taken over all positive or negative whole number values for m, will be seen as a function of the p magnitudes v, and we will designate it by $\vartheta(v_1, v_2, \ldots, v_p)$, such that we have

(1) $$\vartheta(v_1, v_2, \ldots, v_p) = \left(\sum_{-\infty}^{\infty}\right)^p e^{\left(\sum\limits_1^p\right)^2 a_{\mu,\mu'} m_\mu m_{\mu'} + 2\sum\limits_1^p v_\mu m_\mu},$$

where the sums for the exponent are taken over μ and μ', and those of the exterior summation are taken over $m_1, m_2, \ldots, m_p$. For this series to converge, the real part of $\left(\sum\limits_1^p\right)^2 a_{\mu,\mu'} m_\mu m_{\mu'}$ must be essentially negative, which is to say that, represented as the sum of positive or negative squares of linear, real, mutually independent functions of m, it must be composed of p negative squares.

The function ϑ ((jouit)) from this property: that there exist systems of simultaneous increases of values of the p magnitudes v, for which $\log \vartheta$ varies only as a linear function of magnitudes v, and the number of the mutually independent among these systems (which is to say among whom none is a consequence of the others), is $2p$. In effect, in neglecting to write the magnitudes v under the sign of the function ϑ, ((qui n'éprouvent pas de changemenet de valeur)), we have, for $\mu = 1, 2, \ldots, p$,

(2) $$\vartheta = \vartheta(v_\mu + \pi i)$$

and

(3) $$\vartheta = e^{2v_\mu + a_{\mu,\mu}} \vartheta(v_1 + a_{1,\mu}, v_2 + a_{2,\mu}, \ldots, v_p + a_{p,\mu}),$$

as is seen from the following, when in the series ϑ we change the index m_μ into $m_{\mu+1}$, an operation which while not altering the value, makes it take the form in the RHS of (3).

The function ϑ is determined, to a constant factor ((près)), and by these relations, and by the property which it exhibits of being always finite. In effect, following this last property and the relation (2), it is a function of $e^{2v_1}, e^{2v_2}, \ldots, e^{2v_p}$, and is uniform and finite when the values of ν are finite, and, consequently, it is a function developable by a p-ply infinite series of the form

$$\left(\sum_{-\infty}^{\infty}\right)^p A_{m_1,m_2,\ldots,m_p} e^{2\sum\limits_1^p v_\mu m_\mu}$$

with constant coefficients A. But from (3) we derive

$$A_{m_1,\ldots,m_\nu+1,\ldots,m_p} = A_{m_1,\ldots,m_\nu,\ldots,m_p} e^{2\sum\limits_1^p a_{\mu,\nu} m_\mu + a_{\nu,\nu}}$$

and, it follows,

$$A_{m_1,\ldots,m_p} = \text{const.} e^{\left(\sum\limits_1^p\right)^2 a_{\mu,\mu'} m_\mu m_{\mu'}}, \text{ QED.}$$

We can thus employ these properties of the function to define it. The systems of simultaneous increases of the values of magnitudes ν, by the effect of which log ϑ varies as a continuous function, will be called *systems of conjugate moduli of periodicity* of independent variables in this function ϑ.

§ XVIII.

I will now substitute for the p magnitudes $\nu_1, \nu_2, \ldots \nu_p$, p magnitudes $u_1, u_2, \ldots, u_p$, which are always-finite integrals of rational functions of a variable z and of an algebraic function s, $(2p+1)$-ply connected, of this magnitude z, and for the conjugate moduli of periodicity of the magnitudes ν, I will substitute the conjugate moduli of periodicity (which is to say relative to the same cross-cut) of these integrals, in such a way that thus log ϑ is transformed into a function of a *unique* variable z which varies as a linear function of the magnitudes u when s and z return to their same values after a continuous variation of z.

It will first of all come up to demonstrate that such a substitution is possible for any $(2p+1)$-ply function s of z. To this effect, the decomposition of the surface T must be performed through the use of $2p$ cuts $a_1, a_2, \ldots, a_p, b_1, b_2, \ldots, b_p$ coming back upon themselves in a manner that satisfies the following conditions:

When we choose $u_1, u_2, \ldots, u_p$ so that the modulus of periodicity of u_μ, relative to the cut a_μ, be equal to πi, and that, relative to the other cuts ((sections)) a, the moduli of periodicity of u_μ are zero, so, designating by $a_{\mu,\nu}$ the modulus of periodicity of u_μ, relative to the cut b_ν, we will have

$$a_{\mu,\nu} = a_{\nu,\mu},$$

and the real part of

$$\sum_{\mu,\mu'} a_{\mu,\mu'} m_\mu m_{\mu'}$$

must be negative for all the real (whole) values of the p magnitudes m.

§ XIX.

The decomposition of the surface T will not be performed, as we have here, solely by the use of cross-cuts returning back upon themselves, but rather in the following manner. First, perform a cut a_1 returning upon itself and not dividing the surface, then, draw out a cross-cut b_1, starting from the positive side of a_1, and returning to the departure-point on the negative side of this same cut, the which forms a boundary of a *single* piece. Then, we can perform a third cross-cut (when the surface is not yet made simply connected), starting from an arbitrary point on this boundary to join again at any other arbitrary point on the boundary, a point which, consequently, can also be situated on the anterior path of this cross-cut. We will adopt this latter mode of operation; such that this cross-cut is formed by a line a_2 returning on itself and by a portion c_1, run ahead of this line and re-connecting it to the system of preceding cuts. The cross-cut following b_2 will be drawn from the positive side of a_2 to the point of departure on the negative side, and the boundary so formed is still of a *single* piece.

The final ((ultérieure)) decomposition, while necessary, will be again formed anew by two cuts a_2 and b_2 coming from the same points and joined there, and by a line attaching these cuts, c_2, uniting the system of lines a_2 and b_2 to the lines a_3 and b_3. If this procedure is continued until the surface becomes simply connected, we will obtain a network of sections ((sections)) formed by p pairs of two lines, a_1 and b_1, a_2 and b_2, …, a_p and b_p, originating and ending (respectively) in the same point, and by $p – 1$ lines c_1, c_2, …, c_p which attach each pair to the following. The attaching-line c_ν can be drawn from a point of b_ν to a point of $a_{\nu+1}$. The network of sections will thus be regarded as created thus: the $(2\nu - 1)^{\text{th}}$ cross-cut will be formed by the combination of the line $c_{\nu-1}$ and the line a_ν, originating from the extremity of $c_{\nu-1}$ to return to the same place, and the $2\nu^{\text{th}}$ cross-cut will be formed by the line b_ν, originating from the *positive* side of a_ν to return to a_ν on the *negative* side. The boundary of the surface is thus, after an even number of cuts, formed by *one single* piece; after an odd number of cuts, it is formed of *two* pieces.

An everywhere-finite integral w of a rational function of s and z, takes the same value on both sides of the line c. In effect, all the ((antérieur)) boundary is formed by *a single* piece, and, in the integration taken along it and originating from one side of the line c to return on the other side, the integral $\int dw$ is taken twice and in the backwards direction, relative to each element of the section previously made ((Messy!!)). Suh a function is thus everywhere-continuous on the surface T, except ((hormis)) on lines a and b. We will designate by T″ the surface T decomposed in this way by all these lines.

§ XX.

Now let $w_1, w_2, \ldots, w_p$ be similar, mutually independent functions. Let $A_\mu^{(\nu)}$ be the periodic modulus of w_μ relative to the cross-cut a_ν, and $B_\mu^{(\nu)}$ be that relative to cross-cut b_ν. Then the integral $\int w_\mu dw_{\mu'}$, taken positively around the surface T″, is = 0, since the function under the integral sign is everywhere-finite. In this integration, each one of the lines a and b is made once in the positive direction, once in the negative direction, and during the integrations, when these lines serve as the boundary of the domain whose bounding is made in the positive direction, we must take the value on the positive sides for w_μ, which we will designate by w_μ^+ while in the opposite case we must taken its value on the negative sides, which we will designate by w_μ^-.

Consequently, the integral is equal to the sum of all the integrals $\int \left(w_\mu^+ - w_\mu^-\right) dw_{\mu'}$ relative to lines a and b. The b lines lead from the positive side to the negative side of a lines, and it follows that the a lines lead from the negative side to the positive side of the b lines. The integral taken along line a_ν is thus equal to

$$\int A_\mu^{(\nu)} dw_{\mu'} = A_\mu^{(\nu)} \int dw_{\mu'} = A_\mu^{(\nu)} B_{\mu'}^{(\nu)},$$

and the integral taken along b_ν is equal to

$$\int B_\mu^{(\nu)} dw_{\mu'} = -B_\mu^{(\nu)} A_{\mu'}^{(\nu)}.$$

The integral $\int w_\mu dw_{\mu'}$, taken in the positive direction around the surface T″, is therefore equal to

$$\sum_\nu \left(A_\mu^{(\nu)} B_{\mu'}^{(\nu)} - B_\mu^{(\nu)} A_{\mu'}^{(\nu)}\right),$$

and it follows that this sum is equal to zero. This last relation holds for any combination of two functions $w_1, w_2, \ldots, w_p$ and it thus entails $p\,(p-1) / (1 \cdot 2)$ relations among their periodic moduli.[(RH)]

When we take functions u in place of functions w, which is to say that they are chosen in such a way that $A_\mu^{(\nu)} = 0$ for $\mu \neq \nu$, and that $A_\nu^{(\nu)} = \pi i$, these relationships transform into

$$B_{\mu'}^{(\mu)} \pi i - B_\mu^{(\mu')} \pi i = 0$$

or into

$$a_{\mu,\mu'} = a_{\mu',\mu}.$$

§ XXI.

It still remains to be proven that the magnitudes a possess the second property that we have previously found to be necessary.

Suppose that $w = \mu + \nu i$, and, for the modulus of this function relative to cross-cut a_ν, let us suppose $A^{(\nu)} = \alpha_\nu + \gamma_\nu i$, and for the modulus relative to cross-cut b_ν, let us suppose $B^{(\nu)} = \beta_\nu + \delta_\nu i$. Then the integral

$$\int \left(\left(\frac{\partial \mu}{\partial x}\right)^2 + \left(\frac{\partial \mu}{\partial y}\right)^2 \right) dT$$

or

$$\int \left(\frac{\partial \mu}{\partial x}\frac{\partial \nu}{dy} - \frac{\partial \mu}{\partial y}\frac{\partial \nu}{dx} \right) dT$$ [5]

[(RH)] This is like Riemann's habitation dissertation.

[5] This integral expresses the area of the surface which represents (on the w-plane) the totality of values taken by w on the interior of T″. – RIEMANN

relative to T″, is equal to the integral $\int \mu\, d\nu$, taken along the boundary of T″ in the positive direction, and, consequently, it is equal to the sum of the integrals $\int(\mu^+ - \mu^-)d\nu$ relative to the *a* and *b* lines. The integral relative to line a_ν is equal to $\alpha_\nu \int d\nu = \alpha_\nu\delta_\nu$, the integral relative to line b_ν is equal to $\beta_\nu \int d\nu = \beta_\nu\gamma_\nu$, and it follows that

$$\int\left(\left(\frac{\partial\mu}{\partial x}\right)^2 + \left(\frac{\partial\mu}{\partial y}\right)^2\right)dT = \sum_{\nu=1}^{\nu=p}(\alpha_\nu\delta_\nu - \beta_\nu\gamma_\nu).$$

This sum is therefore always positive.

We derive from this the property of magnitudes *a*, which we are showing, in replacing *w* by $u_1m_1 + u_2m_2 + \ldots u_pm_p$. Then

$$A^{(\nu)} = m_\nu\pi i,\; B^{(\nu)} = \sum_\mu a_{\mu,\nu}m_\mu;$$

then a_ν is always equal to zero, and we have

$$\int\left(\left(\frac{\partial\mu}{\partial x}\right)^2 + \left(\frac{\partial\mu}{\partial y}\right)^2\right)dT = -\sum\beta_\nu\gamma_\nu = -\pi\sum m_\nu\beta_\nu,$$

which is to say that the integral is equal to the real part of

$$-\pi\sum_{\mu,\nu} a_{\mu,\nu}m_\mu m_\nu,$$

which expression therefore is positive for all the real values of the magnitudes *m*.

§ XXII.

In the ϑ-series (1),§XVII, let us now replace $a_{\mu,\mu'}$ by the periodic modulus of the function u_μ relative to the cut $b_{\mu'}$, and let us designate by $e_1, e_2, \ldots, e_p$ arbitrary constants. Let us replace ν_μ by $u_\mu - e_\mu$. We then obtain a determined function *z*, uniform for all points of T,

$$\vartheta(u_1 - e_1, u_2 - e_2, \ldots, u_p - e_p),$$

which is continuous and finite except on the *b* lines, and which on the positive side of the line b_ν is $\left(e^{-2(u_\nu - e_\nu)}\right)$ times greater than on the negative side, when we assign the values of the *u* functions on the *b* lines themselves the mean of the values taken on the two sides. The number of points of T′, which is to say the number of pairs of values of *s* and *z* for which this function becomes infinitely small of the first order, can be determined by the consideration of the integral $\int d\log\vartheta$, taken positively around the boundary of T′. In effect, this integral is equal to the number of points in question, multiplied by $2\pi i$. Additionally, this integral is equal to the sum of the integrals $\int(d\log\vartheta^+ - d\log\vartheta^-)$ taken along all the cutting-lines *a*, *b*, and *c*. The integrals taken with respect to lines *a* and *c* are equal to 0, but the integral relative to b_ν is equal to $-2\int du_\nu = 2\pi i$, and therefore the sum of all the integrals is equal to $p2\pi i$. The function ϑ will therefore be infinitely small of the first order on the surface T′ at *p* points which we can designate by $\eta_1, \eta_2, \ldots, \eta_p$.

A positive circuit made by (*s*,*z*) around one of these points increases $\log\vartheta$ by $2\pi i$, but made along the pair of cuts a_ν, b_ν it changes $\log\vartheta$ by $-2\pi i$; consequently, to determine the function $\log\vartheta$ in an everywhere uniform manner, we will perform a cut – from one of these points η along the interior of the surface, la surface, aboutissant chaque fois à une des paires de lignes, la section l_ν partant de η_ν se rapportant à a_ν et b_ν, et cela en la faisant aboutir à l'origine commune de ces lignes, et la fonction sera déterminée comme étant partout continue sur la surface T* formée de la sorte. Elle est alors sur le bord positif des lignes l

It is therefore changed by $-2\pi i$ on the positive side of the l lines, by $g_\nu 2\pi i$ on the positive side of the line a_ν, and by $-2(u_\nu - e_\nu) - h_\nu 2\pi i$ on the positive side of the line b_ν, than it is on the negative sides of the aforesaid lines, g_ν and h_ν designating whole numbers.

The position of the points η and the values of the numbers g and h depend on the magnitudes e, and this dependence can be determined as follows with more precision. The integral $\int \log\vartheta du_\mu$, taken positively around T^* is = 0, as the function $\log\vartheta$ remains continuous on T^*. But this integral is equal to the sum of the integrals $\int(d \log \vartheta^+ - d \log \vartheta^-)du_\mu$, taken along the cutting-lines l, a, b, and c, and we find, in designating the value of u_μ at point η_ν by $\alpha_\mu^{(\nu)}$, that it is equal to

$$2\pi i\left(\sum_\nu \alpha_\mu^{(\nu)} + h_\mu \pi i + \sum_\nu g_\nu a_{\nu,\mu} - e_\mu + k_\mu\right),$$

an expression where k_μ is independent of e, g, h and of the position of points η. This expression is therefore equal to zero.

The magnitude k_μ depends on the choice of the function u_μ, which is solely determined (apart from the additive constant) by the condition of having the periodic modulus πi relative to the cut a_μ, and a periodic modulus of zero for the remaining cuts a. If we take for u_μ a function which exceeds it by a constant c_μ and, if we increase e_μ at the same time by c_μ, the function ϑ remains unchanged and, it follows that the points η and the magnitudes g, h remain unaltered, but the value of u_μ at the point η_ν becomes $\alpha_\mu^{(\nu)} + c_\mu$; consequently k_μ is transformed into $k_\mu - (p-1)c_\mu$ and disappears if we take

$$c_\mu = \frac{k_\mu}{p-1}.$$

We can therefore (as we will finally do), determine the additive constants in the functions u or the initial values in the integrals which express them in such a way, that by substituting $u_\mu - \Sigma\alpha_\mu^{(\nu)}$ for v_μ in $\log\vartheta(v_1, v_2, \ldots, v_p)$ we obtain a function which becomes logarithmically infinite at points η and which, prolonged in a continuous manner on T^*, will have differences on the positive compared to the negative side of: $-2\pi i$ for lines l, 0 for lines a, and $-2\left(u_\nu - \sum_1^p \alpha_\nu^{(\mu)}\right)$ for line b_ν.

To determine these initial values, later on we will employ easier means than those afforded by the expression of k_μ by integrals.

§ XXIII.

If we suppose $(u_1, u_2, \ldots, u_p) \equiv (\alpha_1^{(p)}, \alpha_2^{(p)}, \ldots, \alpha_p^{(p)})$, relative to $2p$ systems of moduli of the functions u (§XV), and, also

$$(v_1, v_2, \ldots, v_p) \equiv \left(-\sum_1^{p-1} \alpha_1^{(\nu)}, -\sum_1^{p-1} \alpha_2^{(\nu)}, \ldots, -\sum_1^{p-1} \alpha_p^{(\nu)}\right),$$

we will have $\vartheta=0$. Reciprocally, if $\vartheta=0$ for $v_\mu=r_\mu$ then $(r_1, r_2, \ldots, r_p)$ is congruent to a system of magnitudes of the form

$$\left(-\sum_1^{p-1} \alpha_1^{(\nu)}, -\sum_1^{p-1} \alpha_2^{(\nu)}, \ldots, -\sum_1^{p-1} \alpha_p^{(\nu)}\right).$$

In effect, if we suppose that $v_\mu = u_\mu - \alpha_\mu^{(p)} + r_\mu$, where η_p is chosen arbitrarily, the function ϑ, first-order infinitely small at η_p, will also be so at $p-1$ other points; and, if we designate them by $\eta_1, \eta_2, \ldots, \eta_{p-1}$, we have

$$\left(-\sum_1^{p-1} \alpha_1^{(\nu)}, -\sum_1^{p-1} \alpha_2^{(\nu)}, \ldots, -\sum_1^{p-1} \alpha_p^{(\nu)}\right) \equiv (r_1, r_2, \ldots, r_p).$$

The function ϑ remains unaltered when we change the magnitudes v into $-v$; in effect, if in the series

$$\vartheta(v_1, v_2, \ldots, v_p)$$

we change the signs of all the indices m, which does not change the value of the series, since $-m_\nu$ takes the same values as m_ν, $\vartheta(v_1, v_2, \ldots, v_p)$ becomes $\vartheta(-v_1, -v_2, \ldots, -v_p)$.

If we arbitrarily take the points $\eta_1, \eta_2, \ldots, \eta_{p-1}$, we will have

$$\vartheta\left(-\sum_1^{p-1} \alpha_1^{(\nu)}, \ldots, -\sum_1^{p-1} \alpha_p^{(\nu)}\right) = 0$$

and consequently, since the function ϑ is even as we have just seen, we will also have

$$\vartheta\left(\sum_1^{p-1} \alpha_1^{(\nu)}, \ldots, \sum_1^{p-1} \alpha_p^{(\nu)}\right) = 0.$$

We can therefore determine the $p-1$ points $\eta_p, \eta_{p+1}, \ldots, \eta_{2p-2}$ in such a way that

$$\left(\sum_1^{p-1} \alpha_1^{(\nu)}, \ldots, \sum_1^{p-1} \alpha_p^{(\nu)}\right) \equiv \left(-\sum_p^{2p-2} \alpha_1^{(\nu)}, \ldots, -\sum_p^{2p-2} \alpha_p^{(\nu)}\right)$$

and consequently that

$$\left(\sum_1^{2p-2} \alpha_1^{(\nu)}, \ldots, \sum_1^{2p-2} \alpha_p^{(\nu)}\right) \equiv (0, \ldots, 0).$$

The position of the $p-1$ last points thus depend on the positions of the $p-1$ first points, in such a way that, with these changing position continuously, we have $\sum_1^{2p-2} d\alpha_\pi^{(\nu)} = 0$ for $\pi = 1, 2, \ldots, p$, and thus (§XVI) the points η are $2p-2$ points for which one of the expressions dw becomes infinitely small of the second order; this comes to mean that, if we designate the value of a pair of magnitudes (s,z) at point η_ν by (σ_ν,ζ_ν), then $(\sigma_1,\zeta_1)\ldots(\sigma_{2p-2},\zeta_{2p-2})$, are the pairs of values associated (§XVI) by the composition of the equation $\varphi = 0$.

If we choose the initial values of the u integrals as we have done here, then we will have

$$\left(\sum_1^{2p-2} u_1^{(\nu)}, \ldots, \sum_1^{2p-2} u_p^{(\nu)}\right) \equiv (0, \ldots, 0),$$

where the summations are taken with respect to all the common roots between the equations $F = 0$ *and* $c_1\varphi_1 + c_2\varphi_2 + \ldots + c_p\varphi_p$ *and different from the pairs of magnitudes* $(\gamma_\rho, \delta_\rho)$ (§VI), *with arbitrary constants c*. ((Poor translation AbelFn.pdf-p.49))

If $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m$ designate m points for which ξ, a rational function of s and z which becomes m-ply first-order infinite, takes the same value, and if $u_\pi^{(\mu)}$, s_μ, z_μ designate the values of u_π, s, z at a point ε_μ, then (§XV)

$$(\sum_1^m u_1^{(\mu)}, \sum_1^m u_2^{(\mu)}, \ldots, \sum_1^m u_p^{(\mu)})$$

are congruent to a constant system of magnitudes $(b_1, b_2, \ldots, b_p)$, which is to say congruent to a system independent of the value of ξ, and we can therefore, for each arbitrary position of a point ε, determine the position of those which remain, in such a way that we have

$$\left(\sum_1^m u_1^{(\mu)}, \ldots, \sum_1^m u_p^{(\mu)}\right) \equiv (b_1, \ldots, b_p).$$

We can therefore take $(u_1 - b_1, \ldots, u_p - b_p)$, when $m = p$, and when $m < p$ we can take

$$\left(u_1 - \sum_1^{p-m} \alpha_1^{(\nu)} - b_1, \ldots, u_p - \sum_1^{p-m} \alpha_p^{(\nu)} - b_p\right)$$

for any arbitrary position of the point (s,z) and the $p - m$ points η, and bring it to be in the form

$$(-\sum_1^{p-1} \alpha_1^{(\nu)}, \ldots, -\sum_1^{p-1} \alpha_p^{(\nu)})$$

in causing one of the points ε to coincide with (s,z), and consequently

$$\vartheta\left(u_1 - \sum_1^{p-m} \alpha_1^{(\nu)} - b_1, \ldots, u_p - \sum_1^{p-m} \alpha_p^{(\nu)} - b_p\right)$$

is equal to 0 for all values whatsoever of pairs of (s,z) and the $p - m$ pairs of (σ_ν,ζ_ν).

§ XXIV.

From the considerations of §XXII results the corollary that any arbitrarily given system of magnitudes $(e_1,\ldots,e_p)$ is always congruent to a single and unique system of values of the form

$$(\sum_1^p \alpha_1^{(\nu)}, \ldots, \sum_1^p \alpha_p^{(\nu)})$$

when the function $\vartheta(u_1 - e_1, \ldots, u_p - e_p)$ does not vanish identically; in effect, it is therefore necessary that the points η be the p points for which this function is equal to zero.

But when the function $\vartheta(u_1^{(p)} - e_1, \ldots, u_p^{(p)} - e_p)$ vanishes for each value of (s_p,z_p), we can write (§XXIII) that

$$(u_1^{(p)} - e_1, \ldots, u_p^{(p)} - e_p) \equiv \left(-\sum_1^{p-1} u_1^{(\nu)}, \ldots, -\sum_1^{p-1} u_p^{(\nu)}\right)$$

and consequently for each value of the pair of magnitudes (s_p,z_p), the pairs $(s_1,z_1)\ldots(s_{p-1},z_{p-1})$ can be determined such that

$$\left(\sum_1^p u_1^{(\nu)}, \ldots, \sum_1^p u_p^{(\nu)}\right) \equiv (e_1, \ldots, e_p),$$

and so it follows that when the position of (s_p,z_p) varies continuously we have for $\pi = 1, 2, \ldots, p$,

$$\sum_1^p du_\pi^{(\nu)} = 0.$$

The p pairs of magnitudes (s_ν, z_ν) are therefore p roots, different from the pairs $(\gamma_\rho,\delta_\rho)$, of an equation $\varphi = 0$, whose coefficients change continuously and in such a way that the $p - 2$ other remaining roots stay constant.

If the values of u_π for these $p - 2$ pairs of values of s and z are denoted by $u_\pi^{(p+1)}$, $u_\pi^{(p+2)}$, $u_\pi^{(2p-2)}$, then

$$\left(\sum_1^{2p-2} u_1^{(\nu)}, \ldots, \sum_1^{2p-2} u_p^{(\nu)}\right) \equiv (0, \ldots, 0),$$

and it follows that

$$(e_1, \ldots, e_p) \equiv \left(-\sum_{p+1}^{2p-2} u_1^{(\nu)}, \ldots, -\sum_{p+1}^{2p-2} u_p^{(\nu)}\right).$$

Reciprocally, when this congruence holds, we have

$$\vartheta(u_1^{(p)} - e_1, \ldots, u_p^{(p)} - e_p) = \vartheta\left(\sum_p^{2p-2} u_1^{(\nu)}, \ldots, \sum_p^{2p-2} u_p^{(\nu)}\right) = 0.$$

An arbitrary given system of values $(e_1, \ldots, e_p)$ *is therefore congruent to a single unique system of values of the form*

$$(\sum_1^p \alpha_1^{(\nu)}, \ldots, \sum_1^p \alpha_p^{(\nu)})$$

when it is not congruent to a system of values of the form

$$(-\sum_1^{p-2}\alpha_1^{(\nu)},\ldots,-\sum_1^{p-2}\alpha_p^{(\nu)})$$

and in the contrary case, it is congruent to an infinity of systems.

Since

$$\vartheta\left(u_1-\sum_1^p\alpha_1^{(\mu)},\ldots,u_p-\sum_1^p\alpha_p^{(\mu)}\right)=\vartheta\left(\sum_1^p\alpha_1^{(\mu)}-u_1,\ldots,\sum_1^p\alpha_p^{(\mu)}-u_p\right),$$

ϑ is a function of each of the p pairs of magnitudes (σ_μ,ζ_μ), just as it is of (s,z). This function of (σ_μ,ζ_μ) will be null for the pair of values (s,z) and for the $p-1$ points which are associated, by the action of the equation $\varphi = 0$, with the $p-1$ remaining pairs (σ,ζ). In effect, if we designate the value of u_π in these points by $\beta_\pi^{(1)}, \beta_\pi^{(2)},\ldots, \beta_\pi^{(p-1)}$, we have

$$\left(\sum_1^p\alpha_1^{(\mu)},\ldots,\sum_1^p\alpha_p^{(\mu)}\right)\equiv\left(\alpha_1^{(\mu)}-\sum_1^{p-1}\beta_1^{(\nu)},\ldots,\alpha_p^{(\mu)}-\sum_1^{p-1}\beta_p^{(\nu)}\right)$$

and then $\vartheta = 0$ when η_μ coincides with one of these points of with the point (s,z).

§ XXV.

Thanks to the previously shown properties of the function ϑ, we obtain the expression of $\log\vartheta$ by the use of integrals of algebraic functions of (s,z), (σ_1,ζ_1), …, (σ_p,ζ_p).

The magnitude

$$\log\vartheta\left(u_1^{(2)}-\sum_1^p\alpha_1^{(\mu)},\ldots\right)-\log\vartheta\left(u_1^{(1)}-\sum_1^p\alpha_1^{(\mu)},\ldots\right)$$

regarded as a function of (σ_μ,ζ_μ) is a function of position of point η_μ, which becomes discontinuous at point ε_1 since it is $-\log(\zeta_\mu-z_1)$, at point ε_2 since it is $-\log(\zeta_\mu-z_2)$; on the positive side of a line joining ε_1 to ε_2, the function is increased by $2\pi i$ and on the positive side of the line b_ν by $2(u_\nu^{(1)}-u_\nu^{(2)})$ above what it is on the negative side of these respective lines, but save on the b lines and the attachment-line going from ε_1 to ε_2, it remains everywhere continuous.

Let us now designate by $\varpi^{(\mu)}(\varepsilon_1,\varepsilon_2)$ an arbitrary function of (σ_μ,ζ_μ) which, except along b lines, is discontinuous in a similar manner, and which, on one of the sides of such a line, is always greater by an equal amount than its value on the other side; this function differs from the preceding (§III) only by a value independent of (σ_μ,ζ_μ), and, consequently, it only differs from

$$\sum_1^p\varpi^{(\mu)}(\varepsilon_1,\varepsilon_2)$$

by a magnitude independent of all the (σ,ζ) and which, consequently, depends only on (s_1,z_1) and (s_2,z_2). Thus $\varpi^{(\mu)}(\varepsilon_1,\varepsilon_2)$ expresses, for $(s,z)=(\sigma_\mu,\zeta_\mu)$, the value of a function $\varpi(\varepsilon_1,\varepsilon_2)$ of §IV, a function whose moduli of periodicity relative to cuts a are equal to 0. If we add a constant c to this function, then

$$\sum_1^p\varpi^{(\mu)}(\varepsilon_1,\varepsilon_2)$$

is increased by *pc*; we can therefore (as will be done in what follows), determine the additive constant in the function $\varpi(\varepsilon_1,\varepsilon_2)$ or the initial value in the integral of the third species which represents this function, such that we have

$$\log\vartheta^{(2)} - \log\vartheta^{(1)} = \sum_1^p \varpi^{(\mu)}(\varepsilon_1,\varepsilon_2).$$

Since ϑ depends on each pair of magnitudes (σ,ζ) in a manner analogous to its dependence upon (s,z), the variation of $\log\vartheta$, when any one of the pairs (s,z), (σ_1,ζ_1), ..., (σ_p,ζ_p) is varied by a finite amount while the rest remain constant, can be expressed as a sum of functions ϖ.

Consequently, we can, by varying each pair (s,z), (σ_1,ζ_1), ..., (σ_p,ζ_p) in succession, express $\log\vartheta$ as a sum of functions ϖ and by $\log\vartheta(0,0,\ldots,0)$ or the value of $\log\vartheta$ for another arbitrary system of values.

The determination of $\log\vartheta(0,0,\ldots,0)$ as a function of the $3p-3$ moduli of periodicity of the system of rational functions of *s* and *z* (§XII), necessitates considerations analogous to those employed by Jacobi in his work on elliptic functions for the determination of $\Theta(0)$. We can succeed in this when, with the aid of the equations

$$4\frac{\partial\vartheta}{\partial a_{\mu,\mu}} = \frac{\partial^2\vartheta}{\partial v_\mu^2} \quad \text{and} \quad 2\frac{\partial\vartheta}{\partial a_{\mu,\mu'}} = \frac{\partial^2\vartheta}{\partial v_\mu\,\partial v_{\mu'}},$$

μ differing from μ', we express in the equation

$$d\log\vartheta = \sum \frac{\partial\log\vartheta}{\partial a_{\mu,\mu'}}\,da_{\mu,\mu'}$$

the derivatives of $\log\vartheta$, taken with respect to the magnitudes *a*, by means of integrals of algebraic functions.

To perform these calculations, a more complete theory of functions which satisfy a linear differential equation with algebraic coefficients seems necessary, a theory which I have a mind to take up next by making use of the methods employed here.

If (s_2,z_2) differs infinitely little from (s_1,z_1), then $\varpi(\varepsilon_1,\varepsilon_2)$ becomes $\partial z_1\, t(\varepsilon_1)$, where $t(\varepsilon_1)$ designates an integral of the second species of a rational function of *s* and *z*, which is discontinuous at ε_1 since there it is $\dfrac{1}{z-z_1}$, and whose moduli of periodicity relative to cuts *a* have the value zero. We find thus that the modulus of periodicity of such an integral relative to the cut b_ν is equal to $2\dfrac{du_\nu^{(1)}}{dz_1}$, and that the constant of integration can be determined such that the sum of the values of $t(\varepsilon_1)$ for the *p* pairs of values $(\sigma_1,\xi_1),\ldots,(\sigma_p,\zeta_p)$ is equal to $\dfrac{\partial\log\vartheta^{(1)}}{dz_1}$.

Thus $\dfrac{\partial\log\vartheta^{(1)}}{d\zeta_\mu}$ is equal to the sum of the values of $t(\eta_\mu)$ for the $(p-1)$ pairs of values associated, by the action of the equation $\varphi = 0$, with the $(p-1)$ pairs of values (σ,ζ) different from (σ_μ,ζ_μ) and for the pair of values (s,z). For

$$\frac{\partial \log \vartheta^{(1)}}{\partial z_1} dz_1 + \sum_1^p \frac{\partial \log \vartheta^{(1)}}{\partial \zeta_\mu} d\zeta_\mu = d \log \vartheta^{(1)},$$

we thus find an expression which was given by Weierstrass [*Crelle's Journal*, v.47, p.300, eqn. (35)] for the case where s is a function of z which is only double-valued.

The properties of $\varpi(\varepsilon_1,\varepsilon_2)$ and $t(\varepsilon_1)$, as functions of (s_1,z_1) and (s_2,z_2), can be derived from the equations

$$\varpi(\varepsilon_1, \varepsilon_2) = \frac{1}{p}\Big(\log \vartheta(u_1^{(2)} - pu_1, \ldots) - \log \vartheta(u_1^{(1)} - pu_1, \ldots)\Big)$$

and

$$t(\varepsilon_1) = \frac{1}{p}\frac{\partial \log \vartheta(u_1^{(1)} - pu_1, \ldots)}{\partial z_1},$$

which are included, as particular cases, in the preceding expressions for $\log\vartheta^{(2)} - \log\vartheta^{(1)}$ and $\dfrac{\partial \log \vartheta^{(1)}}{\partial z_1}$.

§ XXVI.

We can now treat the problem of representing algebraic functions of z in the form of quotients of two products, each of the same number of functions $\vartheta(u_1 - e_1, \ldots,)$, multiplied by powers of magnitudes e^u.

Such an expression, in the case of the crossing of cross-cuts by (s,z), tests the addition of constant factors, and these must be roots of one, when this expression must depend on z algebraically, which is to say, consequently, that when extended in a continuous manner, it must only take a finite number of values for the same z. If all these factors are μ^{th} roots of unity, then the μ^{th} power of this expression is a uniform function, and it follows that it be rational in s and z.

Reciprocally, it can easily be demonstrated that any algebraic function r of z which, extended in a continuous manner along the interior of the surface T′, takes everywhere only *one* value and which, in crossing a cross-cut, is multiplied by a constant factor, can be expressed in a variety of manners as the quotient of two products of functions ϑ and of powers of e^u. Let us designate a value of u_μ for $r = \infty$ by β_μ and for $r = 0$ by γ_μ, and let us draw from each of these points where r is infinite of the first order a line extending to a point where it is infinitely small of the first order, a line in the interior of T′. Now, the function $\log r$ will be everywhere continuous on T, except along these lines. Thus, if $\log r$, is $g_\nu 2\pi i$ greater on the positive side of line b_ν than on the negative side, and differs by $-h_\nu 2\pi i$ on the positive side of a_ν compared to its negative side, then by considering the contour integral (or line integral) $\int \log r \, du_\mu$, we will have

$$\sum \gamma_\mu - \sum \beta_\mu = g_\mu \pi i + \sum_\nu h_\nu a_{\mu,\nu}$$

where $\mu = 1, 2, \ldots, p$ and where g_ν and h_ν, following the preceding remarks, must be rational numbers where the sums of the first member* of the preceding equation must be

* left-hand-side

with respect to all the points where *r* is infinitely small or infinitely great of the first order, in observing that a point where *r* is infinitely small or infinitely great of a higher order must be regarded (§II) as the unification of several points of first order.

When all these points are given, except *p* among them, these last *p* points can always be determined, and that, in general, in a single and unique manner, such that the 2*p* factors $e^{g_\nu 2\pi i}, e^{h_\nu 2\pi i}$ take prescribed values (§§XV,XXIV).

Now, in the expression

$$\frac{P}{Q}e^{-2\sum h_\nu u_\nu},$$

where P and Q each represent a product of the same number of functions $\vartheta(u_1 - \sum \alpha_1^{(\pi)}, \ldots)$ with the same (*s*,*z*) and different (σ,ζ), if pairs of values of *s* and *z* for which *r* becomes infinite are substituted for the pairs (σ,ζ) in the ϑ functions in the denominator, and substitute the pairs of values for which *r* vanishes for the pairs (σ,ζ) in the ϑ functions in the numerator, and if the remaining (σ,ζ) pairs are taken as equal in the denominator as in the numerator, then the logarithm of this expression has the same discontinuities as log *r* in the interior of the surface T′, and exhibits (in the same way as log *r*) in its crossings of the *a* and *b* lines an increase by a constant, purely imaginary magnitude along these lines.

Following *Dirichlet's Principle*, this logarithm differs from log *r* only by a constant, and the expression itself differs from *r* only by a constant factor. As a matter of course this substitution is only admissible when none of the ϑ functions disappears for each of the values of *z*.

This case will occur (§XXIII) if all the pairs of values, for which a uniform function of (*s*,*z*) vanishes, be substituted in the same ϑ function in place of the pairs (σ,ζ).

§ XXVII.

A uniform function, i.e. a rational function of (*s*,*z*), cannot therefore be expressed in the form of a quotient of *two* functions ϑ, multiplied by the powers of magnitudes e^u. But all functions *r* which, for the same pair of values *s* and *z*, take several values and are infinite of the first order only for *p* or fewer pairs of values, are representable in this form; and these functions *r* include all the algebraic functions of *z* representable in this form. We obtain, ignoring constant factors, each of these once and only once, when, in

$$\frac{\vartheta(v_1 - g_1\pi i - \sum_\nu h_\nu a_{1,\nu}, \ldots)}{\vartheta(v_1, \ldots, v_p)} e^{-2\sum_\nu v_\nu h_\nu}$$

rational positive fractions less than 1 are taken for h_ν and g_ν, and v_ν* is replaced by

$$u_\nu - \sum_1^p \alpha_\nu^{(\mu)}.$$

This magnitude is equally an algebraic function of each of the magnitudes ζ, and the principles developed in the preceding paragraph suffice to completely find its algebraic expression by means of magnitudes $z, \zeta_1, \ldots, \zeta_p$.

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* This is vee-sub-nu. It is hard to tell them apart by appearance.

In effect, seen as a function of s,z, when it is extended along the entirety of surface T′ and taking everywhere a *unique* determined value, it becomes first-order infinite for the pairs (σ_1,ζ_1), ..., (σ_p,ζ_p), and, when going from the positive to the negative side of the cut a_ν, it undergoes the addition of the factor $e^{h_\nu 2\pi i}$ and similarly for b_ν by a factor of $e^{-g_\nu 2\pi i}$; each other function of (s,z) which fulfills these same conditions differs from it only by a factor which is independent of (s,z). Considered as a function of (σ_μ,ζ_μ), as it is extended continuously over the entire surface T′, it has everywhere a *unique* determined value, it becomes first-order infinite for the pairs of values (s,z) and for the $(p-1)$ pairs of values $\left(\sigma_1^{(\mu)},\zeta_1^{(\mu)}\right),\dots,\left(\sigma_{p-1}^{(\mu)},\zeta_{p-1}^{(\mu)}\right)$ associated, by the action of the equation $\varphi = 0$, with the $(p-1)$ other remaining pairs of magnitudes (σ,ζ); it undergoes, relative to the cut a_ν, the addition of the factor $e^{-h_\nu 2\pi i}$ and, relative to the cut b_ν, that of the factor $e^{g_\nu 2\pi i}$; any other function of (σ_μ,ζ_μ) which fulfills the same conditions can only differ by a factor independent of (σ_μ,ζ_μ). If we consequently determine an algebraic function

$$f[(s,z);(\sigma_1,\zeta_1),\dots,(\sigma_p,\zeta_p)]$$

of $z, \zeta_1, \dots, \zeta_p$, such that, as being a function of each of these magnitudes, it has these same properties, it differs from the envisioned function only by a factor independent of all the $z, \zeta_1, \dots, \zeta_p$, and will thus be equal to Af, where A designates this factor. To determine this factor A, let us express in f the pairs (σ,ζ) which differ from (σ_μ,ζ_μ) by $\left(\sigma_1^{(\mu)},\zeta_1^{(\mu)}\right),\dots,\left(\sigma_{p-1}^{(\mu)},\zeta_{p-1}^{(\mu)}\right)$, which will give to f the form

$$g\left[(\sigma_\mu,\zeta_\mu);(s,z),\left(\sigma_1^{(\mu)},\zeta_1^{(\mu)}\right),\dots,\left(\sigma_{p-1}^{(\mu)},\zeta_{p-1}^{(\mu)}\right)\right].$$

Plainly, we thus obtain the inverse value of the function to be represented, and, accordingly, an expression which must be equal to $\frac{1}{\mathrm{A}f}$ when, in Ag, we substitute for (σ_μ,ζ_μ) the pair (s,z) and when for the pairs (s,z), $\left(\sigma_1^{(\mu)},\zeta_1^{(\mu)}\right),\dots,\left(\sigma_{p-1}^{(\mu)},\zeta_{p-1}^{(\mu)}\right)$ we substitute the values of pairs (s,z) for which the function to be represented, and accordingly f as well, will be equal to zero.

We obtain in this manner A^2 and, thereafter, A as well, ignoring the sign which can be determined by the direct consideration of the series ϑ in the expression to be represented.

Göttingen, 1857.

END
